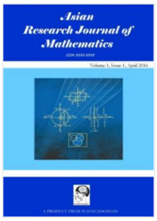

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The Double Auxiliary Equations Method and its Application to Some Nonlinear Evolution Equations

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Authors contributions

This work was carried out in collaboration among all authors. Author AAM Suggested the double auxiliary equations method and drafted the article. Author IMEA Applied the double auxiliary equations method to find the exact solutions of the generalized regularized long wave (RLW) equation. Author AKO Applied the double auxiliary equations method to find the exact solutions of the nonlinear Schrodinger equation. Author LAA Supervised development of work, edited the manuscript and helped to evaluate, revise the manuscript.

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Abstract

Throughout this article, symbolic computation will be used in order to construct a more general exact solutions of the nonlinear evolution equations through a new method called the double auxiliary equations' method, the method represent the study focus of this article. The method has proven applicable and practical through its applications to the generalized regularized long wave (RLW) equation and nonlinear Schrodinger equation.

Keywords: double auxiliary equations method; (RLW) equation ; the nonlinear Schrodinger equation; traveling wave solution.

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1 Introduction

Nonlinear partial differential equations (NPDEs) are used in physics as models to describe many important phenomena in several fields of science, such as plasma physics, nonlinear optics,fluid mechanics, solid state physics, fluid flow,biology, chemistry, economy, and so on. For this very reason, establishing exact traveling wave solutions of NPDEs is essential to properly understand nonlinear phenomena as well as other practical real-life applications.

In the past, a number of methods have been developed to generate analytical solutions of nonlinear partial differential equations. One of these methods are the $\left(\frac{G'}{G}\right)$ expansion method [1,2], the $\left(\frac{G'}{G}, \frac{1}{G}\right)$ expansion method [3], the $\exp\left(-\phi\left(\xi\right)\right)$ expansion method [4,5], the generalized $\exp\left(-\phi\left(\xi\right)\right)$ expansion method [6,7], the $\coth_a\left(\xi\right)$ expansion method [8], the F-expansion method [9], the $\left(\tanh\left(\xi\right)\right)$ expansion method [10], and various other methods [11-15].

In this paper, the double auxiliary equations method is introduced and used to deal with the following two nonlinear differential equations (NPDEs):

(I) The generalized regularized long wave (RLW) equation

$$u_t + u_x + a\left(u^2\right)_x - bu_{xxt} = 0,$$

where a and b are positive constants. RLW equation was first introduced as a model for small amplitude long waves on the surface of water in a channel by Peregrine [16] and later by Benjamin [17].

(II) The nonlinear Schrodinger equation

$$iW_t = -\frac{1}{2}W_{xx} + \delta\left|W\right|^2 W,$$

The nonlinear Schrodinger equation is a nonlinear variation of the Schrodinger equation. It is a field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime[18,19].

The remainder of the paper is organized as follows. Section 2 explains the double auxiliary equations method. Section 3 applies this method for solving the RLW equation. Section 4, applies this method for solving the nonlinear Schrodinger equation. Section 5 concludes the paper.

2 Description of Double Auxiliary Equations Method

Suppose that we have a nonlinear partial differential equation for $u = u(x,t)$ in the form:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...) = 0, \tag{2.1}$$

where P is a polynomial in $u = u(x,t)$ and their various partial derivatives including derivatives.

The main steps of the double auxiliary equations method are as follows:

Step 1. Use the traveling wave transformation:

$$u(x,t) = u(\xi)\ , \qquad \xi = x - vt, \tag{2.2}$$

where v is a non-zero constant to be determined latter, which reduces (2.1) to an (ODE) for $u = u(\xi)$ in the form:

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, ...) = 0. \tag{2.3}$$

Step 2. Balance the highest derivative term with the nonlinear terms in (2.3) to find the value of the positive integer (m). If the value (m) is non-integer one can transform the equation studied.

Step 3. Suppose that the solution of (2.3) can be expressed as follows:

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{h(\xi)}{g(\xi)}\right)^i \tag{2.4}$$

where, $\alpha_i\ (i = 0, \pm1, ..., \pm m)$ are constants to be determined, such that $\alpha_i \neq 0$ and $h(\xi)$ and $g(\xi)$ satisfies the following system of two equations:

$$\begin{cases} \left(\frac{h(\xi)}{g(\xi)}\right)' = A\left(\frac{h(\xi)}{g(\xi)}\right)^2 + B\left(\frac{h(\xi)}{g(\xi)}\right) + C \\ F(x, t, h(\xi), g(\xi), h'(\xi), g'(\xi), ...) = 0, \end{cases} \tag{2.5}$$

where, $F(x, t, h(\xi), g(\xi), h'(\xi), g'(\xi), ...) = 0$, is a differential equation or algebraic equation.

Step 4. Substituting (2.4) into (2.3) and using (2.5), and then setting all the coefficients of $\left(\frac{h(\xi)}{g(\xi)}\right)^i$ of the resulting systems to zero, yields a system of algebraic equations for A, B, C, v and $\alpha_i\ (i = 0, \pm1, ..., \pm m)$.

Step 5. Suppose that the value of the constants A, B, C, k and $\alpha_i\ (i = 0, \pm1, ..., \pm m)$ can be found

by solving the algebraic equations which are obtained in Step 4. Since the general solutions of (2.3) have been well known, substituting A, B, C, v, α_i and the solutions of (2.5) into (2.4), we obtain the exact solutions for Eq. (2.1).

In this paper we will choose the following system

$$\begin{cases} \left(\frac{h(\xi)}{g(\xi)}\right)' = A\left(\frac{h(\xi)}{g(\xi)}\right)^2 + B\left(\frac{h(\xi)}{g(\xi)}\right) + C \\ h'(\xi) = \exp(\xi);\ A, B, C \in \mathbb{R} \end{cases} \tag{2.6}$$

The system (2.6) gives the following solutions:

Family 1. When $\left(B^2 - 4AC\right) > 0$,

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{2C\left(C_1\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\left(B - C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \left(BC_1 - \sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)};\ C_1 \in \mathbb{R} \tag{2.7}$$

Family 2. When $\left(B^2 - 4AC\right) < 0$,

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{2C\left(C_1\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + i\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\left(iB + C_1\sqrt{4AC-B^2}\right)\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + \left(BC_1 - i\sqrt{4AC-B^2}\right)\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)};\ C_1 \in \mathbb{R} \tag{2.8}$$

Family 3. When $\left(B^2 - 4AC\right) = 0$,

$$\left(\frac{h(\xi)}{g(\xi)}\right) = -\frac{2C(C_1\xi + 1)}{B + C_1(B\xi - 2)};\ C_1 \in \mathbb{R} \tag{2.9}$$

Family 4. When $B = 0, AC \neq 0, C > 0,$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = \frac{i\sqrt{(e^{\xi}+C_1)}\sqrt{C}}{\sqrt{-A(e^{\xi}+C_1)}\tan\left(\frac{\left(i\sqrt{-A(e^{\xi}+C_1)}\sqrt{C}\right)\xi - C_2\sqrt{(e^{\xi}+C_1)}}{\sqrt{(e^{\xi}+C_1)}}\right)}; C_1, C_2 \in \mathbb{R} \tag{2.10}$$

Family 5. When $B = 0, AC \neq 0, C < 0,$

$$\left(\frac{h(\xi)}{g(\xi)}\right) = \frac{-\sqrt{(e^{\xi}+C_1)}\sqrt{|C|}}{\sqrt{-A(e^{\xi}+C_1)}\tan\left(\frac{\left(-\sqrt{-A(e^{\xi}+C_1)}\sqrt{|C|}\right)\xi - C_2\sqrt{(e^{\xi}+C_1)}}{\sqrt{(e^{\xi}+C_1)}}\right)}; C_1, C_2 \in \mathbb{R} \tag{2.11}$$

Note that if $A = -1, B = -\lambda, C = -\mu, h(\xi) = G(\xi), g(\xi) = G'(\xi), F = g'(\xi) - h(\xi)$ then the double auxiliary equations method becomes which is the foundation of the known $\left(\frac{G'(\xi)}{G(\xi)}\right)$ expansion method for solving partial differential equations (PDEs) [1,2].

Note that if $A = -k, B = 0, C = k, h(\xi) = Y(\xi), F = g(\xi) - 1$ then the double auxiliary equations method becomes which is the foundation of the known $(\tanh(\xi))$ expansion method for solving partial differential equations (PDEs) [10].

Note that if $A = -1, B = -\lambda, C = -\mu, g(\xi) = \phi(\xi), F = h(\xi) - g(\xi).\exp(-g(\xi))$ then the double auxiliary equations method becomes which is the foundation of the known $\exp(-\phi(\xi))$ expansion method for solving partial differential equations (PDEs) [4,5].

Thus, the above described double auxiliary equations method is the Generalization of the $\left(\frac{G'(\xi)}{G(\xi)}\right)$, $(\tanh(\xi))$ and $\exp(-\phi(\xi))$ methods .

3 The Exact Solution For (RLW) Equation

In this section, we will apply the double auxiliary equations method to find the exact solutions of the generalized regularized long wave (RLW) equation:

$$u_t + u_x + a\left(u^2\right)_x - bu_{xxt} = 0, \tag{3.1}$$

where a and b are positive constants. Suppose that

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{3.2}$$

where v is a constant. Substituting (3.2) into Eq. (3.1), gives the following nonlinear ordinary differential equation:

$$-vu_{\xi} + u_{\xi} + 2auu_{\xi} + bvu_{\xi\xi\xi} = 0, \tag{3.3}$$

Integrating Eq. (3.2) once with respect to ξ and setting the integration constant as zero yields

$$-vu + u + au^2 + bvu_{\xi\xi} = 0. \tag{3.4}$$

Balancing the highest order nonlinear term u^2 and the highest order partial derivative $u_{\xi\xi}$, we get $m + 2 = 2m$, hence $m = 2$. So we can suppose that Eq. (3.4) has the following ansatz:

$$u(\xi) = \alpha_0 + \alpha_1\left(\frac{h(\xi)}{g(\xi)}\right) + \alpha_2\left(\frac{h(\xi)}{g(\xi)}\right)^2 + \alpha_{-1}\left(\frac{h(\xi)}{g(\xi)}\right)^{-1} + \alpha_{-2}\left(\frac{h(\xi)}{g(\xi)}\right)^{-2} \tag{3.5}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}$ are constants and need to be determined, Substituting (3.5) and (2.6) into (3.4), the left-hand side is converted into polynomials in $\left(\frac{h(\xi)}{g(\xi)}\right)^j$, $(j = 0, \pm1, \pm2, ...)$. By collecting each coefficient of the resulting polynomials and setting them to zero, we obtain a set of simultaneous algebraic equations, which are not presented for sake of clarity, for $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}$ and v. Solving these algebraic equations with the help of algebraic software Maple, we obtain:

Case (1).

$$\begin{cases} \alpha_0 = \frac{6bAC}{ab\left(B^2-4AC\right)-a}, \alpha_1 = \frac{6bAB}{ab\left(B^2-4AC\right)-a}, \alpha_2 = \frac{6bA^2}{ab\left(B^2-4AC\right)-a}, \\ \alpha_{-1} = 0, \alpha_{-2} = 0, v = \frac{1}{1-b\left(B^2-4AC\right)} \end{cases} \tag{3.6}$$

Substituting (3.6) into (3.5), we have:

$$u(\xi) = \frac{6bA}{ab(B^2-4AC)-a}\left(C + B\left(\frac{h(\xi)}{g(\xi)}\right) + A\left(\frac{h(\xi)}{g(\xi)}\right)^2\right), \tag{3.7}$$

where $\quad \xi = x - \frac{1}{1-b\left(B^2-4AC\right)}t.$

Consequently, the exact solutions of the generalized regularized long wave (RLW) equation with the help of Eq. (2.7) to Eq. (2.11), are obtained in the following form:

Case (1-1). When $\left(B^2 - 4AC\right) > 0$,

$$\begin{cases} u(\xi) = \frac{6bA}{ab\left(B^2-4AC\right)-a}\left(\begin{array}{c} C - B\left(\frac{2C\left(C_1\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)+\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\left(B-C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)+\left(BC_1-\sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)}\right) \\ +A\left(\frac{2C\left(C_1\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)+\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\left(B-C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)+\left(BC_1-\sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)}\right)^2 \end{array}\right), \\ \xi = x - \frac{1}{1-b\left(B^2-4AC\right)}t; C_1 \in \mathbb{R} \end{cases} \tag{3.5}$$

Case (1-2). When $\left(B^2 - 4AC\right) < 0$,

$$\begin{cases} u(\xi) = \frac{6bA}{ab\left(B^2-4AC\right)-a}\left(\begin{array}{c} C - B\left(\frac{2C\left(C_1\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)+i\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\left(iB+C_1\sqrt{4AC-B^2}\right)\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)+\left(BC_1-i\sqrt{4AC-B^2}\right)\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)}\right) \\ +A\left(\frac{2C\left(C_1\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)+i\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\left(iB+C_1\sqrt{4AC-B^2}\right)\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)+\left(BC_1-i\sqrt{4AC-B^2}\right)\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)}\right)^2 \end{array}\right), \\ \xi = x - \frac{1}{1-b\left(B^2-4AC\right)}t; C_1 \in \mathbb{R} \end{cases} \tag{3.6}$$

Case (1-3). When $\left(B^2 - 4AC\right) = 0$,

$$\begin{cases} u(\xi) = -\frac{6bA}{a}\left(C + B\left(-\frac{2C(C_1\xi+1)}{B+C_1(B\xi-2)}\right) + A\left(-\frac{2C(C_1\xi+1)}{B+C_1(B\xi-2)}\right)^2\right), \\ \xi = x - t \end{cases} \tag{3.7}$$

Case (1-4). When $B = 0, AC \neq 0, C > 0,$

$$\begin{cases} u(\xi) = -\frac{6bA}{a(1+4bAC)} \left(C + A \left(\frac{i\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{C}}{\sqrt{-A\left(e^{\xi}+C_1\right)} \tan\left(\frac{\left(i\sqrt{-A\left(e^{\xi}+C_1\right)\sqrt{C}}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}} \right)} \right)^2 \right), \\ \xi = x + \frac{1}{(1+4bAC)}t; C_1, C_1 \in \mathbb{R} \end{cases} \tag{3.8}$$

Case (1-5). When $B = 0, AC \neq 0, C < 0,$

$$\begin{cases} u(\xi) = -\frac{6bA}{a(4bAC+1)} \left(C + A \left(\frac{\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{|C|}}{\sqrt{-A\left(e^{\xi}+C_1\right)} \tan\left(\frac{\left(-\sqrt{-A\left(e^{\xi}+C_1\right)\sqrt{|C|}}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}} \right)} \right)^2 \right), \\ \xi = x + \frac{1}{(4bAC+1)}t; C_1, C_1 \in \mathbb{R} \end{cases} \tag{3.9}$$

Case (2).

$$\begin{cases} \alpha_0 = -\frac{b\left(B^2+2AC\right)}{a+ab\left(B^2-4AC\right)}, \alpha_1 = 0, \alpha_2 = 0, \\ \alpha_{-1} = \frac{-6bBC}{a+ab\left(B^2-4AC\right)}, \alpha_{-2} = \frac{-6bC^2}{a+ab\left(B^2-4AC\right)}, v = \frac{1}{1+b\left(B^2-4AC\right)} \end{cases} \tag{3.10}$$

Substituting (3.10) into (3.5), we have:

$$u(\xi) = \left(\frac{-b}{a + ab\left(B^2 - 4AC\right)} \right) \left(B^2 + 2AC + 6BC \left(\frac{h(\xi)}{g(\xi)} \right)^{-1} + 6C^2 \left(\frac{h(\xi)}{g(\xi)} \right)^{-2} \right) \tag{3.11}$$

where $\quad \xi = x - \frac{1}{1+b\left(B^2-4AC\right)}t.$

Consequently, the exact solution of the generalized regularized long wave (RLW) equation with the help of Eq. (2.7) to Eq. (2.11), are obtained in the following form:

Case (2-1). When $\left(B^2 - 4AC\right) > 0,$

$$\begin{cases} u(\xi) = \left(\frac{6b}{a+ab\left(B^2-4AC\right)} \right) \left(\begin{array}{l} \left(-\frac{B^2+2AC}{6}\right) \\ +BC \left(\frac{2C\left(C_1 \cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\left(B - C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \left(BC_1 - \sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)} \right)^{-1} \\ +C^2 \left(\frac{2C\left(C_1 \cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\left(B - C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \left(BC_1 - \sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)} \right)^{-2} \end{array} \right), \\ \xi = x - \frac{1}{1+b\left(B^2-4AC\right)}t; C_1 \in \mathbb{R} \end{cases} \tag{3.12}$$

Case (2-2). When $\left(B^2 - 4AC\right) < 0$,

$$\begin{cases} u(\xi) = \left(\frac{6b}{a+ab\left(B^2-4AC\right)}\right) \left(\begin{array}{l} \left(-\frac{B^2+2AC}{6}\right) \\ +BC \left(\frac{2C\left(C_1 \cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + i \sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\left(iB + C_1\sqrt{4AC-B^2}\right) \sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + \left(BC_1 - i\sqrt{4AC-B^2}\right) \cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)} \right)^{-1} \\ +C^2 \left(\frac{2C\left(C_1 \cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + i \sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\left(iB + C_1\sqrt{4AC-B^2}\right) \sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + \left(BC_1 - i\sqrt{4AC-B^2}\right) \cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)} \right)^{-2} \end{array} \right), \\ \xi = x - \frac{1}{1+b\left(B^2-4AC\right)} t; C_1 \in \mathbb{R} \end{cases} \tag{3.13}$$

Case (2-3). When $\left(B^2 - 4AC\right) = 0$,

$$\begin{cases} u(\xi) = \left(\frac{-6bC}{a}\right) \left(A + B\left(-\frac{2C(C_1\xi+1)}{B+C_1(B\xi-2)}\right)^{-1} + C\left(-\frac{2C(C_1\xi+1)}{B+C_1(B\xi-2)}\right)^{-2}\right), \\ \xi = x - t; C_1 \in \mathbb{R} \end{cases} \tag{3.14}$$

Case (2-4). When $B = 0, AC \neq 0, C > 0$,

$$\begin{cases} u(\xi) = \left(\frac{-2bC}{a-4abAC}\right) \left(A + 3C\left(\frac{i\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{C}}{\sqrt{-A\left(e^{\xi}+C_1\right)} \tan\left(\frac{\left(i\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{C}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}\right)^{-2}\right), \\ \xi = x - \frac{1}{1-4bAC} t; C_1, C_2 \in \mathbb{R} \end{cases} \tag{3.15}$$

Case (2-5). When $B = 0, AC \neq 0, C < 0$,

$$\begin{cases} u(\xi) = \left(\frac{-2bC}{a-4abAC}\right) \left(A + 3C\left(\frac{\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{|C|}}{\sqrt{-A\left(e^{\xi}+C_1\right)} \tan\left(\frac{\left(-\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{|C|}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}\right)^{-2}\right), \\ \xi = x - \frac{1}{1-4bAC} t; C_1, C_2 \in \mathbb{R} \end{cases} \tag{3.16}$$

4 The Exact Solution for the Nonlinear Schrodinger Equation

In this section, we will apply the double auxiliary equations method to find the exact solutions of the nonlinear Schrodinger equation.

Let us consider the nonlinear Schrodinger equation:

$$iW_t = -\frac{1}{2} W_{xx} + \delta \left|W\right|^2 W \tag{4.1}$$

We may choose the following traveling wave transformation

$$W(x,t) = u(\xi)\exp(i(\alpha x + \beta t)); \qquad \xi = K(x - \alpha t) \tag{4.2}$$

where K, α and β are constants to be determined later. Eq. (4.1) becomes

$$-\left(\alpha^2 + 2\beta\right)u + K^2 u_{\xi\xi} - 2\delta u^3 = 0 \tag{4.3}$$

By balancing the highest order derivative term $(u_{\xi\xi})$ with the nonlinear term $\left(u^3\right)$ in (4.3), gives $(m = 1)$. Therefore, the double auxiliary equations method allows us to use the solution in the following form:

$$u(\xi) = \alpha_0 + \alpha_1\left(\frac{h(\xi)}{g(\xi)}\right) + \alpha_{-1}\left(\frac{h(\xi)}{g(\xi)}\right)^{-1} \tag{4.4}$$

where $\alpha_0, \alpha_1, \alpha_{-1}$ are constants and need to be determined, Substituting (4.4) and (2.6) into (4.3), the left-hand side is converted into polynomials in $\left(\frac{h(\xi)}{g(\xi)}\right)^j$, $(j = 0, \pm 1, \pm 2, ...)$. By collecting each coefficient of the resulting polynomials and setting them to zero, we obtain a set of simultaneous algebraic equations, which are not presented for sake of clarity, for $\alpha_0, \alpha_1, \alpha_{-1}, \alpha, \beta, K$ and v. Solving these algebraic equations with the help of algebraic software Maple, we obtain:

Case (1).

$$\left\{\begin{array}{c} \alpha_0 = \frac{KB}{2\sqrt{\delta}}, \alpha_1 = 0, \alpha_{-1} = \frac{KC}{\sqrt{\delta}}, \alpha = \alpha, \beta = -\frac{1}{2}\alpha^2 - \frac{K^2}{4}(B^2 - 4AC) \\ K = K, C = C, B = B, A = A, \delta = \delta \end{array}\right. \tag{4.5}$$

Substituting (4.5) into(4.4), and using (4.2) we have :

$$W(\xi) = \frac{K}{\sqrt{\delta}}\left(\frac{B}{2} + C\left(\frac{h(\xi)}{g(\xi)}\right)^{-1}\right) \tag{4.6}$$

where

$$\xi = K(x - \alpha t) \tag{4.7}$$

Consequently,the exact solution of the nonlinear Schrodinger equation (4.1) with the help of Eq. (2.7) to Eq. (2.11) are obtained in the following form:

Case (1-1): When $\left(B^2 - 4AC\right) > 0$.

$$\left\{\begin{array}{l} u(\xi) = \frac{K\sqrt{B^2-4AC}\left(\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + C_1\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{2\sqrt{\delta}\left[\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + C_1\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right]}, \\ \xi = K(x - \alpha t); C_1 \in \mathbb{R} \end{array}\right. \tag{4.8}$$

Case (1-2): When $\left(B^2 - 4AC\right) < 0$.

$$\left\{\begin{array}{l} u(\xi) = \frac{K\sqrt{4AC-B^2}\left(i\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) - C_1\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{2\sqrt{\delta}\left[i\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + C_1\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right]}, \\ \xi = K(x - \alpha t); C_1 \in \mathbb{R} \end{array}\right. \tag{4.9}$$

Case (1-3): When $\left(B^2 - 4AC\right) = 0$.

$$\left\{\begin{array}{l} u(\xi) = \frac{KC_1}{\sqrt{\delta}(C_1\xi + 1)}, \\ \xi = K(x - \alpha t); C_1 \in \mathbb{R} \end{array}\right. \tag{4.10}$$

Case (2-4): When $B = 0, AC \neq 0, C > 0$.

$$\begin{cases} u(\xi) = -\dfrac{iK\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{C}\tan\left(\frac{\left(i\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{C}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}{\sqrt{\delta}\sqrt{\left(e^{\xi}+C_1\right)}}, \\ \xi = K(x-\alpha t); C_1, C_2 \in \mathbb{R} \end{cases} \tag{4.11}$$

Case (2-5): When $B = 0, AC \neq 0, C < 0$.

$$\begin{cases} u(\xi) = \dfrac{CK\sqrt{-A\left(e^{\xi}+C_1\right)}\tan\left(\frac{\left(\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{|C|}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}{\sqrt{\delta}\sqrt{\left(e^{\xi}+C_1\right)}}, \\ \xi = K(x-\alpha t); C_1, C_2 \in \mathbb{R} \end{cases} \tag{4.12}$$

Case (2).

$$\begin{cases} \alpha_0 = \frac{KB}{2\sqrt{\delta}}, \alpha_1 = \frac{AK}{\sqrt{\delta}}, \alpha_{-1} = 0, \alpha = \alpha, \beta = -\frac{1}{2}\alpha^2 - \frac{K^2}{4}(B^2 - 4AC) \\ K = K \end{cases} \tag{4.13}$$

Substituting (4.5) into(4.4), and use (4.2) we have :

$$W(\xi) = \frac{K}{\sqrt{\delta}}\left(\frac{B}{2} + A\left(\frac{h(\xi)}{g(\xi)}\right)\right) \tag{4.14}$$

where

$$\xi = K(x - \alpha t) \tag{4.15}$$

Consequently,the exact solution of the nonlinear Schrodinger equation (4.1) with the help of Eq. (2.7) to Eq. (2.11) are obtained in the following form:

Case (2-1): When $\left(B^2 - 4AC\right) > 0$.

$$\begin{cases} u(\xi) = \frac{KB}{2\sqrt{\delta}} - \dfrac{2AKC\left(C_1\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right)}{\sqrt{\delta}\left[\left(B - C_1\sqrt{B^2-4AC}\right)\sinh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right) + \left(BC_1 - \sqrt{B^2-4AC}\right)\cosh\left(\frac{\sqrt{B^2-4AC}}{2}\xi\right)\right]}, \\ \xi = K(x-\alpha t); C_1 \in \mathbb{R} \end{cases} \tag{4.16}$$

Case (2-2): When $\left(B^2 - 4AC\right) < 0$.

$$\begin{cases} u(\xi) = \frac{KB}{2\sqrt{\delta}} - \dfrac{2AKC\left(C_1\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + i\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right)}{\sqrt{\delta}\left[\left(iB + C_1\sqrt{4AC-B^2}\right)\sin\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right) + \left(BC_1 - i\sqrt{4AC-B^2}\right)\cos\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right]}, \\ \xi = K(x-\alpha t); C_1 \in \mathbb{R} \end{cases} \tag{4.17}$$

Case (2-3): When $\left(B^2 - 4AC\right) = 0$.

$$\begin{cases} u(\xi) = -\frac{KB}{2\sqrt{\delta}} - \frac{2AKC(C_1\xi+1)}{\sqrt{\delta}[B + C_1(B\xi-2)]}, \\ \xi = k(x-\alpha t); C_1 \in \mathbb{R} \end{cases} \tag{4.18}$$

Case (2-4): When $B = 0, AC \neq 0, C > 0$.

$$\begin{cases} u(\xi) = \dfrac{iAK\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{C}}{\sqrt{\delta}\sqrt{-A\left(e^{\xi}+C_1\right)}\tan\left(\frac{\left(i\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{C}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}, \\ \xi = K(x-\alpha t); C_1, C_2 \in \mathbb{R} \end{cases} \tag{4.19}$$

Case (2-5): When $B = 0, AC \neq 0, C < 0$.

$$\begin{cases} u(\xi) = -\dfrac{AK\sqrt{\left(e^{\xi}+C_1\right)}\sqrt{|C|}}{\sqrt{\delta}\sqrt{-A\left(e^{\xi}+C_1\right)}\tan\left(\dfrac{\left(-\sqrt{-A\left(e^{\xi}+C_1\right)}\sqrt{|C|}\right)\xi - C_2\sqrt{\left(e^{\xi}+C_1\right)}}{\sqrt{\left(e^{\xi}+C_1\right)}}\right)}, \\ \xi = K(x-\alpha t); C_1, C_2 \in \mathbb{R} \end{cases} \tag{4.20}$$

5 Conclusion

In this article, a new method called the double auxiliary equations method was proposed where the validity of the method has been tested by applying it successfully to the the RLW equation and the nonlinear Schrodinger equation. It was proved that the double auxiliary equations method is a powerful mathematical technique for finding the exact solutions for the partial differential equations..

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Competing Interests

Authors have declared that no competing interests exist.

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