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More on Symmetric Brothers of a Node in a Perfect Binary Tree

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The paper makes an extensive study on the symmetric brothers of a node in a perfect binary tree. Through proving several new properties of the symmetric brothers of a node, it reveals how the symmetric brothers and the symmetric ancestors distribute on the tree and how they are beneficial for designing a searching algorithm of special purpose. Detail mathematical reasoning and proofs are shown together with concrete examples to demonstrate the mathematical traits. The paper is helpful for designing algorithms in blind search related aspects.

Keywords: Binary tree; ancestor; symmetric brother; distribution; geometric progression.

1 Introduction

In a perfect binary tree, whose definition is seen at page 877 in [1] and as illustrated in Fig. 1, a node might have a father, a grandfather or even ancestors of higher generations, as described by certain entries in books [2-5]. On a non-rooted level of a perfect binary tree, it is known that, a node must have a brother that shares a father with it; it also has cousins that share grandfather or ancestors of higher generations. Binary trees play very important roles in solving various searching problems, e.g., the binary search tree. There are various and plentiful

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literatures on studying binary trees, including traversal algorithms like [6], looking for ancestors like [7-11], locating a descendant of a node like [12-14].

The concept of the symmetric brothers of a node on a level was put forward in [15] because it was helpful to find the symmetric ancestors of a node, as applied in [16] to estimate the bounds of the divisors of certain RSA numbers. This paper following the study of [15], continues investigating the properties of the symmetric brothers of a node in a perfect binary tree. By the research, the paper discloses some more amusing properties of the symmetric brothers and shows them.

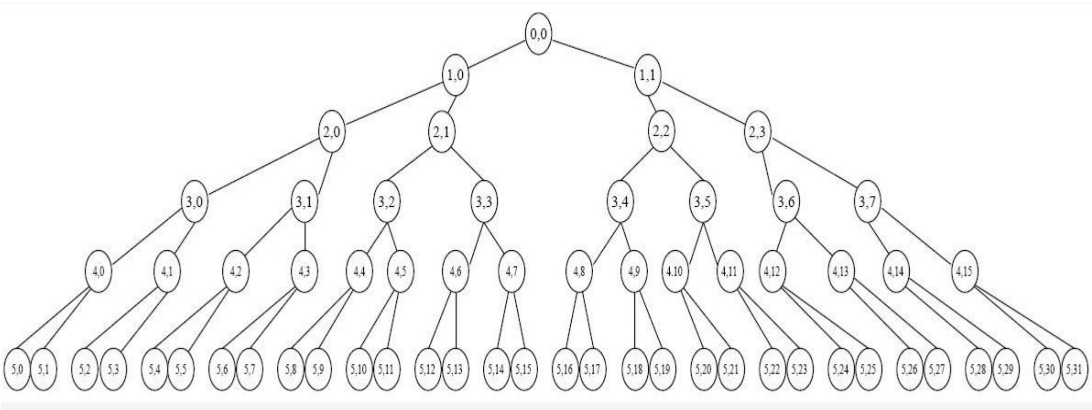

Fig. 1. A perfect binary tree from level 0 to level 5

2 Preliminaries

2.1 Definitions and notations

A perfect binary tree *T*, which was defined at page 877 in [1], is a binary tree with all leaf nodes at the same depth and all the internal nodes have degree 2. Definitions related with the father, the ancestors and the descendants of a node can be found by certain entries in textbooks [2-5]. In this paper, a node of a binary tree refers to either a vertex or a leaf, and a brother means an inbred brother node unless particularly mentioned. For convenience, symbol $N_{(k,j)}$ is the node at the position *j* on level *k* of *T*, where $k \geq 0$ and $0 \leq j \leq 2^k - 1$. The index of the levels by default begins with zero and index of the positions also by default begins with zero. An ancestor means a direct ancestor (namely, not the father's cousins and so forth) unless particularly mentioned. Symbol $T_{N_{(k,j)}}$ denotes a subtree whose root is $N_{(k,j)}$ and symbol $N_{(i,\omega)}^{N_{(k,j)}}$ denotes the node at the position ω on level *i* of $T_{N_{(k,j)}}$. Node $N_{(i,\omega)}^{N_{(k,j)}}$ and node $N_{(i,2^i-1-\omega)}^{N_{(k,j)}}$ are geometrically symmetric on level *i* with $i > 0$ thus they are called symmetric nodes. Accordingly, the path that connects $N_{(k,j)}$ and $N_{(i,\omega)}^{N_{(k,j)}}$ is said to be symmetric to the path that connects $N_{(k,j)}$ and $N_{(i,2^i-1-\omega)}^{N_{(k,j)}}$. The path from $N_{(i,\omega)}^{N_{(k,j)}}$ to $N_{(k,j)}$ and to $N_{(i,2^i-1-\omega)}^{N_{(k,j)}}$ is called a symmetric path. Nodes on the same level and on a symmetric path are symmetric brothers. Use symbol X Ë Y to express *X* and *Y* are symmetric. On a level $k > 0$, the number of nodes between node $N_{(k,s)}^X$ and $N_{(k,t)}^X$ is defined to be the distance from $N_{(k,s)}^X$ to $N_{(k,t)}^X$. Statement that *A* is *d* away from *B* means *d* is the distance between node *A* and *B*. Symbol $X \in l(T_N)$ means node *X* is in the left branch of T_N while symbol $X \in r(T_N)$ means node *X* is in the right branch of T_N.

Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number *x* that satisfies inequality $x-1 < \lfloor x \rfloor \leq x$, or equivalently $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Symbol mod is the modulo operation and expression $r = a \bmod b$ means $a \equiv r (\bmod b)$.

2.2 Lemmas

Lemma 1 [1]. Let $N_{(m,\alpha)}$ be the node at position α on level m of a perfect binary tree T with $m>0$ and $0 \le \alpha \le 2^m - 1$; then the direct ancestors of $N_{(m,\alpha)}$ are calculated by $N_{(m-i,\sigma(i,\alpha))}$, where $\sigma(i,\alpha) = \left\lfloor \frac{\alpha}{2^i} \right\rfloor (i>0)$. The level i with $(i \ge 0)$ of $T_{N_{(m,\alpha)}}$ is the level $m+i$ of T and $N_{(i,\omega)}^{N_{(m,\alpha)}} = N_{(m+i,2^i\alpha+\omega)}$.

Lemma 2 [15]. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k>0$; then there are k symmetric paths connecting $N_{(k,j)}$ and its symmetric nodes. In another word, $N_{(k,j)}$ has k symmetric brothers on level k in ***T*.**

Lemma 3 [15]. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k>0$. If $N_{(k,j)}$ is a left node, then all its symmetric brothers must be right ones; or vice versa, if $N_{(k,j)}$ is a right node, then all its symmetric brothers must be left ones.

Lemma 4 [15]. Let $N_{(k,j)}$ be a node in a perfect binary tree T with $k>0$ and $A_1, A_2, ..., A_k$ be the father, the grandfather and the so-forth ancestors of $N_{(k,j)}$ respectively; then $N_{(k,j)}$ is the node $N_{(1,j \bmod 2^1)}^{A_1}$ in T_{A_1}, the node $N_{(2,j \bmod 2^2)}^{A_2}$ in T_{A_2}, and so forth, the node $N_{(i,j \bmod 2^i)}^{A_i}$ in T_{A_i} until $i=k$. Consequently, its symmetric brothers in T_{A_1} , T_{A_2} ,…, T_{A_i} ,…, T_{A_k} are respectively the nodes $N_{(1,2^1-1-j \bmod 2^1)}^{A_1}$ $N_{(2,2^2-1-j \bmod 2^2)}^{A_2},...,N_{(i,2^i-1-j \bmod 2^i)}^{A_i},...,$ and $N_{(k,2^k-1-j \bmod 2^k)}^{A_k}$; or equivalently,

$$N^{A_1}_{(1,2\left\lfloor \frac{j}{2} \right\rfloor+1-j)}, N^{A_2}_{(2,2^2\left\lfloor \frac{j}{2^2} \right\rfloor+2^2-1-j)}, ..., N^{A_i}_{(i,2^i\left\lfloor \frac{j}{2^i} \right\rfloor+2^i-1-j)}, ..., N^{A_k}_{(k,2^k\left\lfloor \frac{j}{2^k} \right\rfloor+2^k-1-j)}$$

If in terms of *T*, they are

$$N_{(k,2\left\lfloor \frac{j}{2} \right\rfloor+1-j \bmod 2)}, N_{(k,2^2\left\lfloor \frac{j}{2^2} \right\rfloor+2^2-1-j \bmod 2^2)}, ..., N_{(k,2^i\left\lfloor \frac{j}{2^i} \right\rfloor+2^i-1-j \bmod 2^i)}, ..., N_{(k,2^k\left\lfloor \frac{j}{2^k} \right\rfloor+2^k-1-j \bmod 2^k)}$$

or equivalently

$$N_{(k,2^{1+1}\left\lfloor \frac{j}{2} \right\rfloor+1-j))}, N_{(k,2^{2+1}\left\lfloor \frac{j}{2^2} \right\rfloor+2^2-1-j)}, ..., N_{(k,2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor+2^i-1-j)}, ..., N_{(k,2^{k+1}\left\lfloor \frac{j}{2^k} \right\rfloor+2^k-1-j)}$$

Lemma 5 [17]. Properties of the floor functions with real numbers *x* and *y*, integers *m*, *n* and *k*

(**P1**) $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(**P2**) $\lfloor x \rfloor - \lfloor y \rfloor - 1 \le \lfloor x-y \rfloor \le \lfloor x \rfloor - \lfloor y \rfloor < \lfloor x \rfloor - \lfloor y \rfloor + 1$

(**P13**). $x \le y \Rightarrow \lfloor x \rfloor \le \lfloor y \rfloor$

(**P14**) $\lfloor x \pm n \rfloor = \lfloor x \rfloor \pm n$.

(**P16**) $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor, x \in \boldsymbol{Z} \\ -\lfloor x \rfloor - 1, x \notin \boldsymbol{Z} \end{cases}$

(**P17**) $\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x+1}{2} \right\rfloor = \lfloor x \rfloor$

(**P32**) $\alpha \lfloor x \rfloor - 1 < \lfloor \alpha x \rfloor < \alpha(\lfloor x \rfloor + 1)$; particularly $n\lfloor x \rfloor \le \lfloor nx \rfloor \le n(\lfloor x \rfloor + 1) - 1$. Taking n=2 yields $2\lfloor x \rfloor \le \lfloor 2x \rfloor \le 2\lfloor x \rfloor + 1$

(**P41**) For integers $n \ge 0$ and $\alpha \ge 0$, it holds

$$\left\lfloor \frac{-n-1}{2^{\alpha}} \right\rfloor = -1 - \left\lfloor \frac{n}{2^{\alpha}} \right\rfloor$$

3 Main Results with Proofs

Proposition 1. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k>0$ and $0 \le j \le 2^k - 1$; suppose $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ are its symmetric brothers; then

$$N_{(k,j)} \in l(T) \Rightarrow B_{(k,1)} \in l(T), ..., B_{(k,k-1)} \in l(T), B_{(k,k)} \in r(T)$$

and

$$N_{(k,j)} \in r(T) \Rightarrow B_{(k,1)} \in r(T), ..., B_{(k,k-1)} \in r(T), B_{(k,k)} \in l(T).$$

Proof. Let $A_1, A_2, ..., A_k$ be the k ancestors of $N_{(k,j)}$; then by definition, A_k is the root of T and $B_{(k,k)}$ is the symmetric brother of $N_{(k,j)}$ in T_{A_k} . Thereby, $N_{(k,j)} \in l(T) \Rightarrow B_{(k,k)} \in r(T)$ and $N_{(k,j)} \in r(T) \Rightarrow B_{(k,k)} \in l(T)$ **.** Since $A_1, A_2, ..., A_{k-1}$ **are** all in the same branch of *T* as $N_{(k,j)}$ is, the symmetric brothers in $T_{A_1}, T_{A_2}, ..., T_{A_{k-1}}$ are respectively sure in the same branch of *T* as $N_{(k,j)}$ is.

Example 1. Pick in Fig. 1 a node, say the node $N_{(5,8)} = B_0$; it has 5 ancestors: $N_{(4,4)} = A_1, N_{(3,2)} = A_2$, $N_{(2,1)} = A_3, N_{(1,0)} = A_4$ and $N_{(0,0)} = A_5$. Fig.2 intuitionally depicts the distributions as follows

$$\begin{aligned}
&B_1 = N_{(5,9)} \in A_1 \in l(A_5),\\
&B_2 = N_{(5,11)} \in A_2 \in l(A_5),\\
&B_3 = N_{(5,15)} \in A_3 \in l(A_5),\\
&B_4 = N_{(5,7)} \in A_4 \in l(A_5),\\
&B_5 = N_{(5,23)} \in r(A_5).
\end{aligned}$$

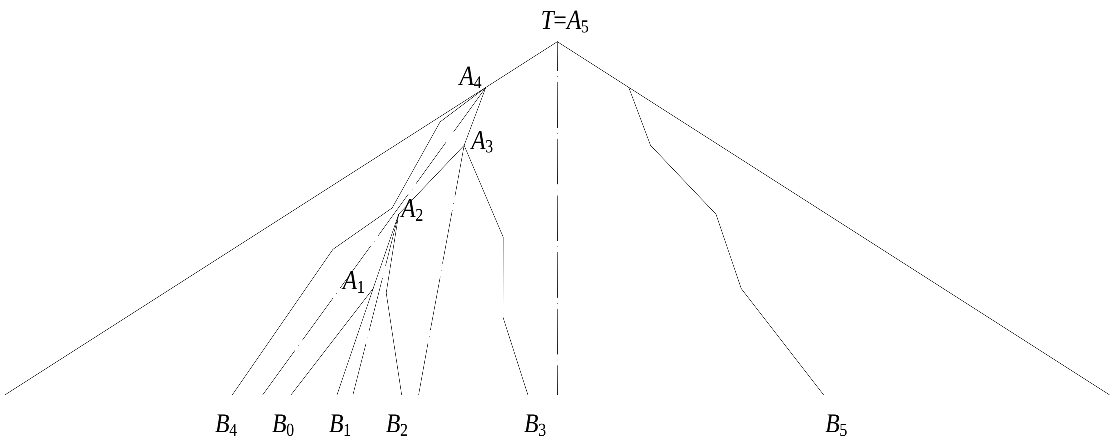

Fig. 2. Distribution of symmetric brothers on a level

Remark 1. For convenience, the pole of $N_{(k,j)}$ in Proposition 1 is called a *pivot*; it is the base node to calculate its symmetric brothers.

Proposition 2. Let $N_{(k,j)} = B_{(k,0)}$ be a node of a perfect binary tree T with $k > 0$, $0 \le j \le 2^k - 1$ and $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ being its symmetric brothers. Take an arbitrary one in $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ as a pivot, say $B_{(k,i)}$ with $1 \le i \le k$; then $N_{(k,j)} (= B_{(k,0)})$ is one of the symmetric brothers of $B_{(k,i)}$.

Proof. Let $A_1, A_2, ..., A_k$ be the direct ancestors of $N_{(k,j)} (= B_{(k,0)})$; then A_i is the common ancestor of $N_{(k,j)}$ and $B_{(k,i)}$. Thereby, by definition, $N_{(k,j)} \in T_{A_i}$ and it is symmetric to $B_{(k,i)}$.

Proposition 3. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k > 0$ and $0 \le j \le 2^k - 1$; suppose A_ω with $1 < \omega < k$ is an ancestor of $N_{(k,j)}$; define the symmetric brothers of A_ω to be symmetric ancestors of $N_{(k,j)}$ respectively; then $N_{(k,j)}$ has $k - \omega$ symmetric ancestors on level ω of *T*.

Proof. By definition, A_k , the root of *T*, is on level 0 of *T* and A_1 is on level $k - 1$. This yields a relationship described in the following Table 1.

Table 1. Ancestors and their levels

Ancestor	A_k	A_{k-1}	…	A_ω	…	A_2	A_1
On level	0	1	…	k-ω	…	k-2	k-1

Accordingly, A_ω lies on level ω and has $k - \omega$ ancestors from its father $A_{\omega+1}$ to A_k and thus has $k - \omega$ symmetric brothers on the level $k - \omega$ of T by Lemma 2. By definition $N_{(k,j)}$ has the $k - \omega$ symmetric brothers as its symmetric ancestors.

Example 2. Seen in Fig. 3, *k*=5 and $\omega = 2$; $A_\omega = A_2$ is an ancestor of node $N_{(k,j)}$ on level 3. The three symmetric brothers of A_ω are symmetric ancestors of $N_{(k,j)}$.

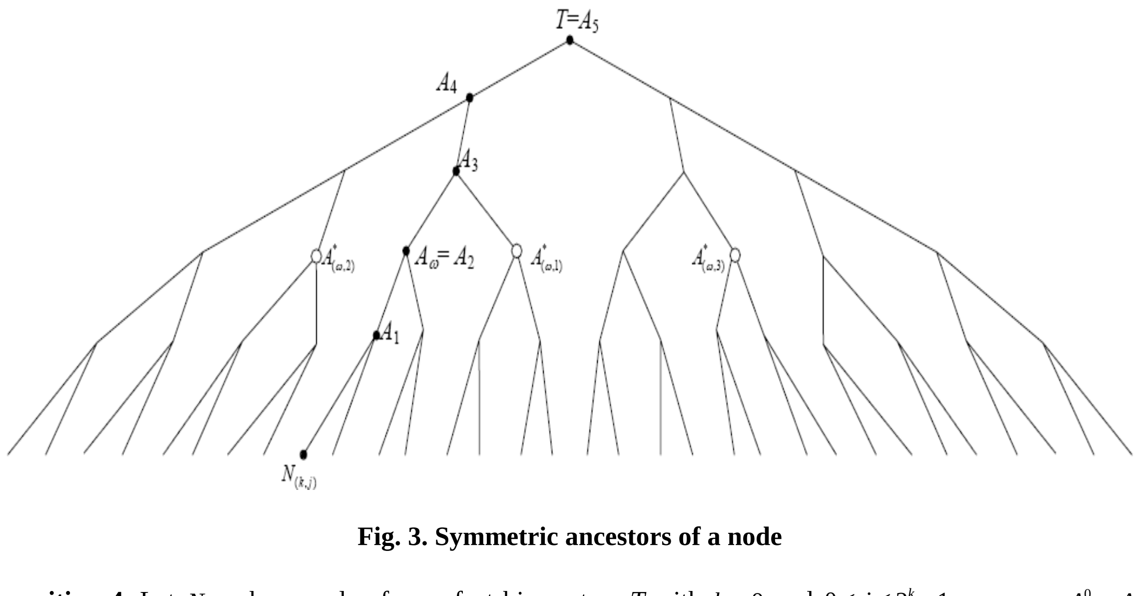

Fig. 3. Symmetric ancestors of a node

Proposition 4. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k > 0$ and $0 \le j \le 2^k - 1$; suppose $A_\omega^0 = A_\omega$ is the ancestor of $N_{(k,j)}$ on level $k - \omega$ with $1 < \omega < k$; denote $A_\omega^1, A_\omega^2, ..., A_\omega^{k-\omega}$ to be the $k - \omega$ symmetric ancestors of $N_{(k,j)}$ respectively; then among the k symmetric brothers $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ of $N_{(k,j)}$, the precedent ω ones $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,\omega)}$ are descendants of A_ω^0 and each of the rest ones is subordinate to one of the symmetric ancestors $A_\omega^1, A_\omega^2, ..., A_\omega^{k-\omega}$ respectively, namely, $B_{(k,\omega+i)} \in T_{A_\omega^i}$ with $i = 1, 2, ..., k - \omega$.

Proof. By Lemma 4, $B_{(k,1)}, B_{(k,2)}, \ldots$ and $B_{(k,\omega)}$ are respectively descendants of $A_1, A_2, \ldots$, and A_ω. Since $A_1, A_2, \ldots$ and $A_{\omega-1}$ are direct descendants of $A_\omega^0 = A_\omega$, it is sure that $B_{(k,1)}, B_{(k,2)}, \ldots$ and $B_{(k,\omega)}$ are descendants of A_ω^0. Consequently, the rest $k-\omega$ ones, $B_{(k,\omega+1)}, B_{(k,\omega+2)}, \ldots, B_{(k,k)}$ are out of $T_{A_\omega^0}$. Note that, A_ω^0 lies on level $k-\omega$ of T and it has $k-\omega$ symmetric brothers. Now take an arbitrary $B_{(k,i)}$ with $\omega+1 \le i \le k$ as an example, we will show $B_{(k,\omega+i)} \in T_{A_\omega^i}$. In fact,

by Lemma 1, the ancestors of $N_{(k,j)}$ on levels $k-\omega$ is

$$A_\omega = N_{(k-\omega, \left\lfloor \frac{j}{2^\omega} \right\rfloor)}.$$

By Lemma 4, the i^{th} symmetric brother of A_ω is

$$A_\omega^i = N_{(k-\omega, 2^{i+1}\left\lfloor \frac{j}{2^{i+\omega}} \right\rfloor + 2^i - 1 - \left\lfloor \frac{j}{2^\omega} \right\rfloor)}.$$

Considering the symmetric brothers, $B_{(k,\omega+1)}, B_{(k,\omega+2)}, \ldots$, and $B_{(k,k)}$, of $N_{(k,j)}$, it holds by Lemma 4

$$B_{(k,\omega+i)} = N_{(k, 2^{\omega+i+1}\left\lfloor \frac{j}{2^{\omega+i}} \right\rfloor + 2^{\omega+i} - 1 - j)}, 1 \le i \le k.$$

Hence its ancestor on level $k-\omega$, the level ω levels away from level *k*, is

$$N_{\left(k-\omega, \left\lfloor \frac{2^{\omega+i+1}\left\lfloor \frac{j}{2^{\omega+i}} \right\rfloor + 2^{\omega+i} - 1 - j}{2^\omega} \right\rfloor\right)} = N_{(k-\omega, 2^{i+1}\left\lfloor \frac{j}{2^{\omega+i}} \right\rfloor + 2^i + \left\lfloor \frac{-1-j}{2^\omega} \right\rfloor)}.$$

By Lemma 5(P41), it knows A_ω^i is the ancestor of $B_{(k,\omega+i)}$ on level $k-\omega$. Accordingly, the proposition holds.

Corollary 1. Let $N_{(k,j)}$ be a node of a perfect binary tree *T* with $k > 0$ and $0 \le j \le 2^k - 1$; suppose $A_\omega^0 = A_\omega$ is the direct ancestor of $N_{(k,j)}$ on level $k-\omega$ of *T*, $A_\omega^1, A_\omega^2, \ldots$, and $A_\omega^{k-\omega}$ are symmetric ancestors; then

$$A_\omega = N_{(k-\omega, \left\lfloor \frac{j}{2^\omega} \right\rfloor)}$$

and

$$A_\omega^i = N_{(k-\omega, 2^{i+1}\left\lfloor \frac{j}{2^{i+\omega}} \right\rfloor + 2^i - 1 - \left\lfloor \frac{j}{2^\omega} \right\rfloor)}, i = 1, 2, \ldots, k-\omega.$$

Proof. Simply applying Lemma 4 to calculate the symmetric brothers of A_ω on level $k-\omega$ of *T*.

Proposition 4*. Let $N_{(k,j)}$ be a node of a perfect binary tree *T* with $k > 0$ and $0 \le j \le 2^k - 1$; suppose $A_\omega^0 = A_\omega$ is the ancestor of $N_{(k,j)}$ on level $k-\omega$ with $1 < \omega < k$. If a node *X* on level $k-\omega$ of *T* is not a symmetric ancestor of $N_{(k,j)}$, then $N_{(k,j)}$ has no symmetric brother lying in the subtree T_X.

Proof. This is an inverse and negative proposition of Proposition 4.

Example 3. Seen in Fig. 4, node 37 is not a symmetric brother of node 33 that is an ancestor of node 271; then T_{37} does not contain a node that is a symmetric brother of the node 271.

Proposition 5. Let $N_{(k,j)}$ be a node of a perfect binary tree T with $k>0$ and $0 \le j \le 2^k - 1$; suppose $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ are its symmetric brothers; let $N_{(k,j)} = B_{(k,0)}$ and $p_0, p_1, p_2, ..., p_k$ be the positions of $B_{(k,0)}, B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ on level k of T, namely, $B_{(k,i)} = N_{(k,p_i)}$; then

$$A_\omega = N_{(k-\omega,\theta)}$$

where

$$\theta = \left\lfloor \frac{p_0 + p_\omega}{2^{\omega+1}} \right\rfloor.$$

Proof. By Lemma 1, it holds

$$A_\omega = N_{(k-\omega, \left\lfloor \frac{j}{2^\omega} \right\rfloor)}.$$

Note that

$$p_0 = j, p_\omega = 2^{\omega+1}\left\lfloor \frac{j}{2^\omega} \right\rfloor + 2^\omega - 1 - j, \omega = 1, 2, ..., k.$$
$$\Rightarrow \frac{p_0 + p_\omega}{2^{\omega+1}} = \left\lfloor \frac{j}{2^\omega} \right\rfloor + \frac{2^\omega - 1}{2^{\omega+1}}.$$
$$\Rightarrow \left\lfloor \frac{p_0 + p_\omega}{2^{\omega+1}} \right\rfloor = \left\lfloor \frac{j}{2^\omega} \right\rfloor.$$

Hence the proposition holds.

Proposition 6. Let $N_{(k,j)} = B_{(k,0)}$ be a node of a perfect binary tree T with $k>0$ and $0 \le j \le 2^k - 1$; suppose $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ are its symmetric brothers; let $p_1, p_2, ..., p_k$ be the positions of $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ on level k of T, namely, $B_{(k,i)} = N_{(k,p_i)}$; then

$$| p_{i+1} - p_i | = 2^i, i = 1, 2, ..., k-1.$$

Proof. By Lemma 4,

$$p_i = 2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor + 2^i - 1 - j, i = 1, 2, ..., k.$$

Direct calculation yields

$$p_{i+1} - p_i = 2^{i+2}\left\lfloor \frac{j}{2^{i+1}} \right\rfloor + 2^{i+1} - 2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor - 2^i$$
$$= 2^{i+2}\left\lfloor \frac{j}{2^{i+1}} \right\rfloor - 2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor + 2^i$$
$$= 2^{i+1}(2\left\lfloor \frac{j}{2^{i+1}} \right\rfloor - \left\lfloor \frac{j}{2^i} \right\rfloor) + 2^i.$$

Let $D = 2\left\lfloor \frac{j}{2^{i+1}} \right\rfloor - \left\lfloor \frac{j}{2^i} \right\rfloor$; then by Lemma 4(P32), $n\lfloor x \rfloor \le \lfloor nx \rfloor \le n(\lfloor x \rfloor + 1) - 1$

$$-1 = \left\lfloor \frac{2j}{2^{i+1}} \right\rfloor - 2 + 1 - \left\lfloor \frac{j}{2^i} \right\rfloor \le D \le \left\lfloor \frac{2j}{2^{i+1}} \right\rfloor - \left\lfloor \frac{j}{2^i} \right\rfloor \le 0.$$

That is

$$-1 \le D \le 0.$$

Consequently

$$p_{i+1} - p_i = \begin{cases} 2^i, D = 0 \\ -2^i, D = -1 \end{cases}.$$

Example 4. Take in Fig. 1 node $N_{(5,13)}$ that leads to $k = 5$ and $j = 13$. The ancestors of $N_{(5,13)}$ are $A_1 = N_{(4,6)}$, $A_2 = N_{(3,3)}$, $A_3 = N_{(2,1)}$, $A_4 = N_{(1,0)}$ and $A_5 = N_{(0,0)}$. In T_{A_1}, $N_{(5,13)} = N^{A_1}_{(1,13 \bmod 2)} = N^{A_1}_{(1,1)}$ and its symmetric brother is $N^{A_1}_{(1,2^1-1-13 \bmod 2)} = N^{A_1}_{(1,0)} = N_{(5,2\left\lfloor \frac{13}{2} \right\rfloor + 2^1 - 1 - 13 \bmod 2)} = N_{(5,12)} = B_{(5,1)}$; in T_{A_2}, $N_{(5,13)} = N^{A_2}_{(2,13 \bmod 2^2)} = N^{A_2}_{(2,1)}$ and its symmetric brother is $N^{A_2}_{(2,2^2-1-13 \bmod 2^2)} = N^{A_2}_{(2,2)} = N_{(5,2^2 \times \left\lfloor \frac{13}{2^2} \right\rfloor + 2^2 - 1 - 13 \bmod 2^2)} = N_{(5,14)} = B_{(5,2)}$; in T_{A_3}, $N_{(5,13)} = N^{A_3}_{(3,13 \bmod 2^3)} = N^{A_3}_{(3,5)}$ and its symmetric brother is $N^{A_3}_{(3,2^3-1-13 \bmod 2^3)} = N^{A_3}_{(3,2)} = N_{(5,2^3 \times \left\lfloor \frac{13}{2^3} \right\rfloor + 2^3 - 1 - 13 \bmod 2^3)} = N_{(5,10)} = B_{(5,3)}$; in T_{A_4}, $N_{(5,13)} = N^{A_4}_{(4,13 \bmod 2^4)} = N^{A_4}_{(4,13)}$ and its symmetric brother is $N^{A_4}_{(4,2^4-1-13 \bmod 2^4)} = N^{A_4}_{(4,4)} = N_{(5,2^4 \times \left\lfloor \frac{13}{2^4} \right\rfloor + 2^4 - 1 - 13 \bmod 2^4)} = N_{(5,2)} = B_{(5,4)}$; in T_{A_5}, $N_{(5,13)} = N^{A_5}_{(5,13 \bmod 2^5)} = N^{A_5}_{(5,13)}$ and its symmetric brother is $N^{A_5}_{(5,2^5-1-13 \bmod 2^5)} = N^{A_5}_{(5,18)} = N_{(5,2^5 \times \left\lfloor \frac{13}{2^5} \right\rfloor + 2^5 - 1 - 13 \bmod 2^5)} = N_{(5,18)} = B_{(5,5)}$. Accordingly, $p_1 = 12, p_2 = 14, p_3 = 10, p_4 = 2, p_5 = 18$, which yields $p_2 - p_1 = 2,$ $p_3 - p_2 = -4, p_4 - p_3 = -8$ and $p_5 - p_4 = 16$.

Proposition 6*. Let $N_{(k,j)} = B_{(k,0)}$ be a node of a valuated binary tree *T* with $k > 0$ and $0 \le j \le 2^k - 1$, which was defined in [18]; suppose $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ are its symmetric brothers; let $p_1, p_2, ..., p_k$ be the positions of $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ on level *k* of *T*, namely, $B_{(k,i)} = N_{(k,p_i)}$; then

$$| B_{(k,i+1)} - B_{(k,i)} | = 2^{i+1}, i = 1, 2, ..., k-1$$

and

$$B_{(k,i)} - B_{(k,0)} = 2^{i+2} \left\lfloor \frac{j}{2^i} \right\rfloor + 2^{i+1} - 2 - 4j, i = 1, 2, ..., k,$$

satisfying $-2^{i+1} \le B_{(k,i)} - B_{(k,0)} \le 2^{i+1} - 2$.

Particularly,

$$B_{(k,k)} - B_{(k,0)} = 2^{k+1} - 2 - 4j.$$

Proof. By Lemma 4, $p_s = 2^{s+1}\left\lfloor \frac{j}{2^s} \right\rfloor + 2^s - 1 - j \ (s = 1, 2, ..., k)$. Note that,

$$B_{(k,s+1)} - B_{(k,s)} = 2(p_{s+1} - p_s) = \pm 2^{s+1}, s = 1, 2, ..., k-1.$$

It leads to

$$|B_{(k,i+1)} - B_{(k,i)}| = 2^{i+1}, i = 1, 2, ..., k-1,$$

and

$$B_{(k,0)} = N_{(k,j)}, B_{(k,1)} = N_{(k,2^{1+1}\left\lfloor \frac{j}{2} \right\rfloor + 1 - j))}, B_{(k,2)} = N_{(k,2^{2+1}\left\lfloor \frac{j}{2^2} \right\rfloor + 2^2 - 1 - j)}, ..., B_{(k,s)} = N_{(k,2^{s+1}\left\lfloor \frac{j}{2^s} \right\rfloor + 2^s - 1 - j)}, ..., B_{(k,k)} = N_{(k,2^{k+1}\left\lfloor \frac{j}{2^k} \right\rfloor + 2^k - 1 - j)}.$$

To know the relationship between $B_{(k,0)}$ and $B_{(k,s)} (s = 1, 2, ..., k)$, direct calculation yields

$$B_{(k,s)} - B_{(k,0)} = N_{(k,2^{s+1}\left\lfloor \frac{j}{2^s} \right\rfloor + 2^s - 1 - j)} - N_{(k,j)} = 2^{s+2}\left\lfloor \frac{j}{2^s} \right\rfloor + 2^{s+1} - 2 - 4j.$$

By Lemma 5(P32), $4j + 1 - 2^{s+2} \le 2^{s+2}\left\lfloor \frac{j}{2^s} \right\rfloor \le 4j$; it follows

$$-2^{s+1} - 1 \le B_{(k,s)} - B_{(k,0)} \le 2^{s+1} - 2.$$

Since $B_{(k,s)}$ and $B_{(k,0)}$ are both odd, it holds

$$-2^{s+1} \le B_{(k,s)} - B_{(k,0)} \le 2^{s+1} - 2.$$

Since $B_{(k,k)} = N_{(k,2^{k+1}\left\lfloor \frac{j}{2^k} \right\rfloor + 2^k - 1 - j)}$ and $0 \le j \le 2^k - 1$, it results in

$$B_{(k,k)} = N_{(k,2^k - 1 - j)}.$$

Consequently,

$$B_{(k,k)} - B_{(k,0)} = 2(2^k - 1 - j) - 2j = 2^{k+1} - 2 - 4j.$$

Example 5. Pick in Fig. 4 a node, say the node $N_{(5,7)} = B_0 = 271$; it has 5 ancestors: $N_{(4,3)} = A_1 = 175, N_{(3,1)} = A_2 = 67, N_{(2,0)} = A_3 = 33, N_{(1,0)} = A_4 = 17$ and $N_{(0,0)} = A_5 = 9$. Obviously, it is seen with Fig. 4,

$$B_1 = N_{(5,6)} = 269,$$
$$B_2 = N_{(5,4)} = 265,$$
$$B_3 = N_{(5,0)} = 257,$$
$$B_4 = N_{(5,8)} = 273,$$
$$B_5 = N_{(5,24)} = 305.$$

It can be tested that, $p_1 = 6, p_2 = 4, p_3 = 0, p_4 = 8$ and $p_5 = 24$ match to $|p_{i+1} - p_i| = 2^i$, which states in Proposition 5. It can also be seen that, $|B_{i+1} - B_i| = 2^{i+1}$ and particularly

$$-2^{1+1} < \underline{B_1 - B_0 = 2^{1+2}\left\lfloor \frac{7}{2^1} \right\rfloor + 2^{1+1} - 2 - 4\times 7} = -2 < 2^{1+1} - 2 = 2,$$

$$-2^{2+1} < \underline{B_2 - B_0 = 2^{2+2}\left\lfloor \frac{7}{2^2} \right\rfloor + 2^{2+1} - 2 - 4\times 7} = 6 < 2^{2+1} - 2 = 6,$$

$$-2^{3+1} < \underline{B_3 - B_0 = 2^{3+2}\left\lfloor \frac{7}{2^3} \right\rfloor + 2^{3+1} - 2 - 4\times 7} = -14 < 2^{3+1} - 2 = 14,$$

$$-2^{4+1} < \underline{B_4 - B_0 = 2^{4+2}\left\lfloor \frac{7}{2^4} \right\rfloor + 2^{4+1} - 2 - 4\times 7} = 2 < 2^{4+1} - 2 = 30,$$

$$-2^{5+1} < \underline{B_5 - B_0 = 2^{5+2}\left\lfloor \frac{7}{2^5} \right\rfloor + 2^{5+1} - 2 - 4\times 7} = 34 < 2^{5+1} - 2 = 62.$$

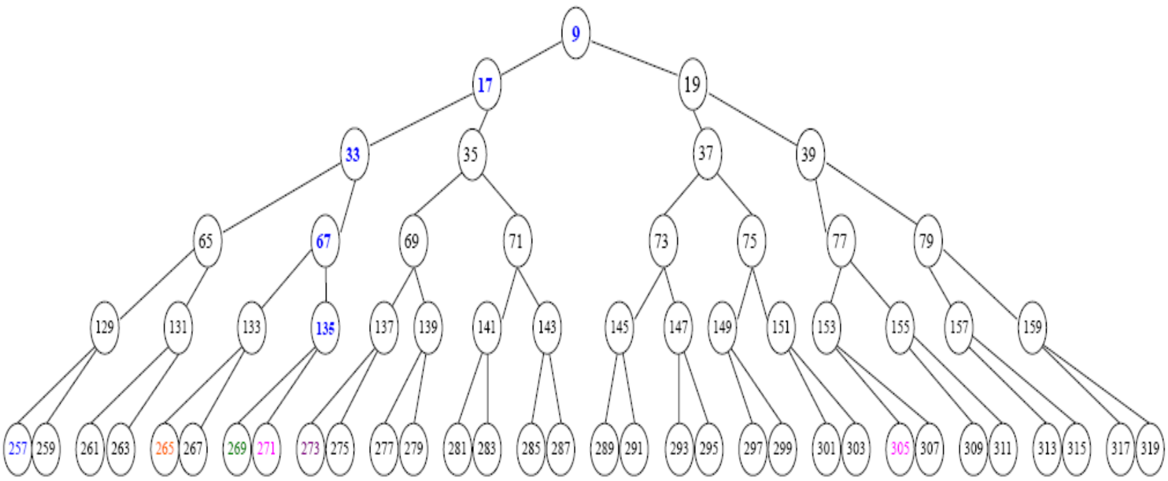

Fig. 4. A T_9 valuated binary tree

Corollary 2. Let $N_{(k,0)}$ be the leftmost node on level k of a perfect binary tree T with $k > 0$; suppose $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ are its symmetric brothers with $p_1, p_2, ..., p_k$ being their positions respectively; then $p_{i+1} - p_i$ ($i = 1, 2, ..., k-1$) form an ascending power series by 2's power.

Proof. $N_{(k,0)}$ being the leftmost node on level k means $p_1 < p_2 < ... < p_k$. By Proposition 4, it knows

$$p_1 = 1, p_2 = 3, p_3 = 7, p_4 = 15, ..., p_i = 2^i - 1, ..., p_k = 2^k - 1.$$

Proposition 7. Let $N_{(k,j)} = B_{(k,0)}$ be a node of a perfect binary tree T with $k > 0$, $0 \le j \le 2^k - 1$ and $B_{(k,1)}, B_{(k,2)}, ..., B_{(k,k)}$ being its symmetric brothers; let $B^*_{(k,1)}, B^*_{(k,2)}, ..., B^*_{(k,k)}$ be symmetric brothers of $N_{(k,2^k-1-j)} = B^*_{(k,0)}$; then it holds in T

$$B^*_{(k,i)} \text{ Ë } B_{(k,i)}, i = 0, 1, ..., k.$$

Proof. The case $i = 0$ is sure. For the cases $1 \le i \le k$ By Lemma 4

$$
\begin{aligned}
B^*_{(k,i)} &= N_{\left(k,2^{i+1}\left\lfloor \frac{2^k-1-j}{2^i} \right\rfloor + 2^i - 1 - (2^k - 1 - j)\right)} \\
&= N_{\left(k,2^{i+1}\left\lfloor 2^{k-i} + \frac{-1-j}{2^i} \right\rfloor + 2^i - 2^k + j\right)} = N_{\left(k,2^{k+1} + 2^{i+1}\left\lfloor \frac{-1-j}{2^i} \right\rfloor + 2^i - 2^k + j\right)} \\
&= N_{\left(k,2^k + 2^{i+1}\left\lfloor \frac{-1-j}{2^i} \right\rfloor + 2^i + j\right)} = N_{\left(k,2^k + 2^{i+1}\left(-1-\left\lfloor \frac{j}{2^i} \right\rfloor\right) + 2^i + j\right)} \\
&= N_{\left(k,2^k - 2^i - 2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor + j\right)} = N_{\left(k,2^k - 1 - \left(2^{i+1}\left\lfloor \frac{j}{2^i} \right\rfloor + 2^i - 1 - j\right)\right)} \\
&\Rightarrow B^*_{(k,i)} \text{ Ë } B_{(k,i)}
\end{aligned}
$$

Example 6. Take in Fig. 1 node $N_{(5,13)}$ that leads to $k=5$ and $j=13$. Then the symmetric brothers of $N_{(5,13)}$ are $N_{(5,12)}, N_{(5,14)}, N_{(5,10)}, N_{(5,2)}, N_{(5,18)}$ and the symmetric brothers of $N_{(5,2^5-1-13)} = N_{(5,18)}$ are $N_{(5,19)}$, $N_{(5,17)}, N_{(5,21)}, N_{(5,29)}$ and $N_{(5,13)}$. It can see

$$
\begin{aligned}
&N_{(5,13)} \text{ Ë } N_{(5,18)} \\
&N_{(5,12)} \text{ Ë } N_{(5,19)} \\
&N_{(5,14)} \text{ Ë } N_{(5,17)} \\
&N_{(5,10)} \text{ Ë } N_{(5,21)} \\
&N_{(5,2)} \text{ Ë } N_{(5,29)}
\end{aligned}
$$

Corollary 3. For an integer $k>0$ take two nodes in a perfect binary tree *T*, $N_{(k,0)}$ and $N_{(k,2^k-1)}$ with their symmetric brothers respectively $B_{(k,0)} = N_{(k,p_0)} = N_{(k,0)}, B_{(k,1)} = N_{(k,p_1)}, B_{(k,2)} = N_{(k,p_2)}, ..., B_{(k,k)} = N_{(k,p_k)}$ and $B^*_{(k,1)} = N_{(k,q_1)}, B^*_{(k,2)} = N_{(k,q_2)}, ..., B^*_{(k,k)} = N_{(k,q_k)} = N_{(k,2^k-1)}$; then

$$
\begin{aligned}
&p_0 = 0 < p_1 = 1 < p_2 = 3 < p_3 = 7 < ... < p_{k-1} = 2^{k-1} - 1 < p_k = 2^k - 1 \\
&q_k = 0 < q_{k-1} = 2^{k-1} < ... < q_2 = 2^k - 4 < q_1 = 2^k - 2 < q_0 = 2^k - 1
\end{aligned}
$$

Proof. (Omitted)

Remark 2. Corollary 2 and Corollary 3 describe the distribution of the symmetric brothers of two end-nodes on a level of *T*, as illustrated in Fig. 5. It can be seen from Corollary 3 that, taking $N_{(k,0)}$ to be the pivot results in $N_{(k,2^{k-1}-1)}$ being a symmetric brother at the rightmost position in the left branch of *T* while taking $N_{(k,2^k-1)}$ to be the pivot results in $N_{(k,2^{k-1})}$ being a symmetric brother at the leftmost position in the right branch of *T*. $k-1$ ones from the *k* symmetric brothers of the pivot $N_{(k,0)}$ are in the left branch of *T* while the *k*th one, $N_{(k,2^k-1)}$, is in the right branch of *T*. Likewise, $k-1$ ones from the *k* symmetric brothers of pivot $N_{(k,2^{k-1})}$ are in the right branch of *T* while the *k*th one, $N_{(k,0)}$, is in the left branch of *T*. A pivot closer to the left has more symmetric brothers on its right side while a pivot closer to the right has more symmetric brothers on its left side.

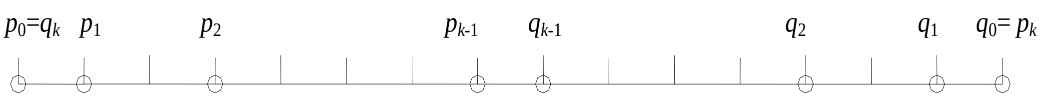

Fig. 5. Distribution of the symmetric brothers of two end-nodes

Corollary 4. On level k of a perfect binary tree T, by means of constructing symmetric brothers, it can always obtain an ordered sequence of nodes among which the maximal distance between two adjacent ones is in an expected range.

Proof. Let $A_1, A_2, ..., A_k$ be the direct ancestors of $N_{(k,0)}$ and take $N_{(k,0)}$ as a pivot to obtain $B_{(k,0)} = N_{(k,p_0)} = N_{(k,_0)}, B_{(k,1)} = N_{(k,p_1)}, B_{(k,2)} = N_{(k,p_2)}, ..., B_{(k,k)} = N_{(k,p_k)}$; then by Proposition 1 $B_{(k,k)} = N_{(k,p_k)}$ is the rightmost node in the right branch of T, and by Proposition 6 and Corollary 2, the distance from $B_{(k,k-1)} = N_{(k,p_{k-1})}$ to $B_{(k,k)} = N_{(k,p_k)}$ is $2^{k-1} - 1$ and it is the maximal distance between two adjacent symmetric brothers, the second maximal distance is $2^{k-2} - 1$ that is from $B_{(k,k-2)} = N_{(k,p_{k-2})}$ to $B_{(k,k-1)} = N_{(k,p_{k-1})}$. Now take $B_{(k,k-1)} = N_{(k,p_{k-1})}$ to be the pivot to obtain k-1 symmetric brothers of $N_{(k,p_{k-1})}$ in $T_{A_{k-1}}$; then Proposition 1 and Corollary 3, the k-1 symmetric brothers of $N_{(k,p_0)}$ together with the k-1 symmetric brothers of $N_{(k,p_{k-1})}$ symmetrically distribute from $N_{(k,p_0)}$ to $N_{(k,p_{k-1})}$ and the maximal distance between two adjacent nodes is $2^{k-3} - 1$. Keep this process going on by α times, it can obtain a maximal distance between two adjacent nodes by $2^{k-\alpha-1} - 1$. By symmetric property, so the result is with the right branch of T.

Example 7. Take in Fig.6 the node 257 as a pivot to obtain it symmetric brothers 259,263,271,287 and 319, which are marked with red color on the bottom. The distance from 287 to 319 is $15 = 2^4 - 1$ and that from 271 to 287 is $7 = 2^3 - 1$. Then take 287 as a pivot and obtain in T_{17} its symmetric brothers, 285,281,273,257, which are marked with blue color. The two results are combined into one by 257,259, 263, 271, 273, 281, 285, 287 and 319 with that $2^2 - 1$ is the maximal distance between two adjacent nodes. Making the result symmetric on the whole level is surely an expectation.

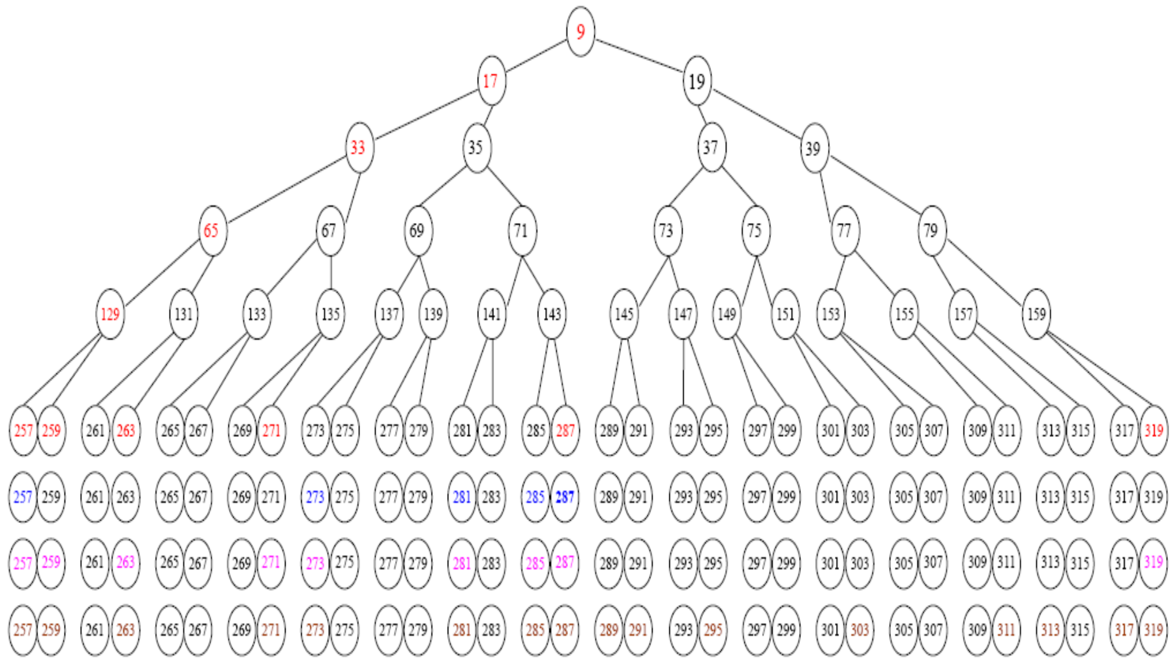

Fig. 6. Symmetric brothers control the maximal distance between two adjacent nodes

4 Conclusion and Expectation

The symmetric brothers of a node reveal an amusing relationship of subordinations among nodes of a perfect binary tree. On the one hand, the distribution of the symmetric brothers of a node demonstrates a characteristic like the geometric progression, as Corollary 2 shows; on the other hand, the distribution exhibits a kind of concentration with singularity: symmetric brothers tend to lie around the pivot while there exists a subtree that does not contain one symmetric brother, as Proposition 3* states. Such distribution is helpful to subdivide a big search into small ones for certain purpose. For example, when we want to find an objective number in a large interval, we can pick a number as a pivot and find its symmetric brothers to be the subdivision elements to subdivide the big interval into small ones. Such a subdivision can be of high efficiency for certain searches because it is something like Fibonacci distribution. We will show the application of the subdivision in the coming paper. We also hope young generation to join this research.

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Competing Interests

Author has declared that no competing interests exist.

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