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Existence of Positive Solution For a Fourth-order Differential System

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, we investigate the existence of positive solutions for the singular fourth-order differential system

$$u^{(4)} = \varphi u + f(t, u, u''), \qquad 0 < t < 1,$$
$$-\varphi'' = \mu g(t, u, u''), \qquad 0 < t < 1,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0,$$
$$\varphi(0) = \varphi(1) = 0;$$

where $\mu > 0$ is a constant, and the nonlinear terms f, g may be singular with respect to the time and space variables. By fixed point theorem in cones, the existence is established for singular differential system. The results obtained herein generalize and improve some known results including singular and non-singular cases.

Keywords: Positive solutions; fixed point theorem; singular solutions; bending of an elastic beam; cone; boundary value problem; existence; multiplicity.

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1 Introduction

It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{1.2}$$

Existence of solutions for problem (1.1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal et.al. [10,11,12] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [13] studied the existence of positive solutions for the second-order differential system by using the fixed point theorem of cone expansion and compression.

In this paper we shall discuss the existence of positive solutions for the fourth-order boundary value problem

$$\begin{gathered} u^{(4)} = \varphi u + f(t, u, u''), \quad 0 < t < 1 \\ -\varphi'' = \mu g(t, u, u''), \ 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{gathered} \tag{1.3}$$

where μ is a positive parameter and $f(t, u, v) : (0, 1) \times [0, \infty) \times (-\infty, 0] \longrightarrow (0, \infty)$ is continuous. In fact as we will see below one could consider in Section 2 and 3 $f(t, u, v) \leq f_1(t)\, f_2(t, u, v)$ with $f_2(t, u, v) : [0, 1] \times [0, \infty) \times (-\infty, 0] \longrightarrow (0, \infty)$ and $f_1 : (0, 1) \to (0, +\infty)$ is continuous provided

$$\int_0^1 \int_0^1 K(\tau, \tau)\, K(\tau, s)\, f_1(s) ds\, d\tau < +\infty;$$

here K is as defined in Section 2. Moreover, our hypotheses allow but do not require $g(t, u, v) : [0, 1] \times (0, \infty) \times (-\infty, 0) \to [0, \infty)$ to be singular at $v = 0$ and $u = 0$.

2 Preliminaries

Let $Y = C[0, 1]$ and

$$Y_+ = \{u \in Y : u(t) \geq 0, \ t \in [0, 1]\}.$$

It is well known that Y is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$. We denote the norm $\|u\|_2$ by

$$\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}.$$

It is easy to show that $C^2[0, 1]$ is complete with the norm $\|u\|_2$ and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Suppose that $K(t, s)$ is the Green function associated with

$$-u'' = f(t), \quad u(0) = u(1) = 0, \tag{2.1}$$

which is explicitly expressed by

$$K(t, s) = \left\{ \begin{array}{ll} t(1 - s), & 0 \leq t \leq s \leq 1 \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{array} \right\}$$

We need the following lemmas.

Lemma 1. $K(t,s)$ has the following properties:

(i) $K(t,s) > 0, \forall t, s \in (0,1)$;
(ii) $K(t,s) \leq K(s,s), \forall t, s \in [0,1]$;
(iii) $K(t,s) \geq K(t,t)K(s,s), \forall t, s \in [0,1]$;
(iv) $|K(t_1,s) - K(t_2,s)| \leq 2|t_1 - t_2|$, for all $t_1, t_2, s \in [0,1]$

Proof. It can be easily seen that (i), (ii) and (iii) are satisfied. Next, we check that (iv) is satisfied. In fact, for $t_1 \leq t_2 \leq s$, or $s \leq t_1 \leq t_2$, it is easy to know that $|K(t_1,s) - K(t_2,s)| \leq |t_1 - t_2|$.

Similarly, for $t_1 \leq s \leq t_2$, we have

$$|K(t_1,s) - K(t_2,s)| \leq |s(1-t_2) - t_1(1-s)| \leq |s(t_1 - t_2) + s - t_1| \leq 2|t_1 - t_2| .$$

This proves that (iv) is satisfied.

This finishes the proof. □

We will investigate the existence of positive solutions for problem (1.3) by the following fixed point theorem of cone expansion and compression of norm type:

Lemma 2 ([14], Theorem 2.3.4.)**.** Let E be a real Banach space and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subset of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $Q : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ be a completely continuous operator such that either

(i) $\|Qu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Qu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or
(ii) $\|Qu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Qu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.
Then Q has a fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Firstly, we will transform the problem (1.3) into a new form.
The boundary value problem

$$-\varphi'' = \mu g(t, u(t), u''(t)), \quad \varphi(0) = \varphi(1) = 0,$$

can be solved by using the Green's function, namely,

$$\varphi(t) = \mu \int_0^1 K(t,s) g(s, u(s), u''(s)) ds, \quad 0 < t < 1. \tag{2.2}$$

Thus inserting (2.2) into the first equation of (1.3), we have

$$u^{(4)} = \mu u(t) \int_0^1 K(t,s) g(s, u(s), u''(s)) ds + f(t, u, u'')$$
$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2.3}$$

Now we consider the existence of a positive solution of (2.3). The function $u \in C^4(0,1) \cap C^2[0,1]$ is a positive solution of (2.3), if $u \geq 0$, $t \in [0,1]$, and $u \neq 0$.

Then the solution of (2.3) can be expressed as

$$u(t) = \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int_0^1 K(s,v) g(v, u(v), u''(v))\, dv\, ds\, d\tau +$$

$$+\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,f(s,u(s),u''(s))\,ds\,d\tau \tag{2.4}$$

and the second-order derivative u'' can be expressed by

$$u''(t) = -\mu\int_0^1 K(t,s)\,u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))\,dv\,ds-$$

$$-\int_0^1 K(t,s)\,f(s,u(s),u''(s))\,ds. \tag{2.5}$$

Set

$$P = \{u\in C^2[0,1] : u(0)=u(1)=0, u(t)\geq K(t,t)\,\|u\|_0\,, -u''(t)\geq K(t,t)\,\|u''\|_0\,,$$

$$t\in[0,1]\,\}.$$

Note P is a cone in $C^2[0,1]$. For $R>0$, write $B_R=\{u\in C^2[0,1] : \|u\|_2 < R\}$.

We now define a mapping $T : P\to C^2[0,1]$ by

$$Tu(t) = \mu\int_0^1\int_0^1 K(t,\tau)\,K(\tau,s)\,u(s)\int_0^1 K(s,v)\,g(v,u(v),u''(v))\,dv\,ds\,d\tau+$$

$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f(s,u(s),u''(s))\,ds\,d\tau. \tag{2.6}$$

It is easy to see that if $u\in P$ than

$$-u''(t)\geq\sigma\,\|u''\|_0\,,\quad t\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{2.7}$$

where $\sigma=\frac{3}{16}$.

Lemma 3. Let $u\in P$. Then the following relations hold:

(a) $(Tu)(t)\geq K(t,t)\,\|Tu\|_0$ for $t\in[0,1]$, and
(b) $-(Tu)''(t)\geq K(t,t)\,\|Tu''\|_0$ for $t\in[0,1]$.

Proof. For simplicity we denote

$$I=\mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)q(v)dvdsd\tau+$$

$$+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)h(s)dsd\tau,$$

$$J=\mu\int_0^1 K(s,s)u(s)\int_0^1 K(s,v)q(v)dvds+\int_0^1 K(s,s)h(s)ds,$$

and

$$q(v)=g(v,u(v),u''(v)),\quad h(s)=f(s,u(s),u''(s)).$$

From Lemma 1 it is easy to see that

$$K(t,t)I\leq Tu(t)\leq I\quad\text{and } t\in[0,1] \tag{2.8}$$

$$K(t,t)J\leq -(Tu)''(t)\leq J,\qquad t\in[0,1] \tag{2.9}$$

Using (2.8-2.9), we have

$$\|Tu\|_0 \leq I, \text{ and } \left\|-(Tu)''\right\|_0 \leq J,$$

hence

$$(Tu)(t) \geq K(t,t)\,\|Tu\|_0 \text{ for } t \in [0,1] \text{ and}$$

$$-(Tu)''(t) \geq K(t,t)\,\left\|Tu''\right\|_0 \text{ for } t \in [0,1].$$

This finishes the proof. □

Throughout this paper, we assume additionaly that the function $f(t,u,v)$ satisfies
(H1)

$$f(t,u,v) \leq f_1(t)f_2(u+|v|), \qquad t \in (0,1), \qquad u \in R^+, \;\; v \in R^-,$$

where $f_1 : (0,1) \to (0,+\infty)$ and $f_2 : [0,+\infty) \to (0,+\infty)$ is continuous, $R^+ = [0,+\infty)$, $R^- = (-\infty,0]$.

Moreover the function $g(t,u,v) : [0,1] \times (0,+\infty) \times (-\infty,0) \to [0,+\infty)$ satisfies
(H2) There exists an $a > 0$ such that $g(t,u,v)$ is nonincreasing in $u \leq a$ and $|v| \leq a$ for each fixed $t \in [0,1]$ i.e. if $-a \leq v_2 \leq v_1 < 0$ and $0 < u_1 \leq u_2$ then $g(t,u_1,v_1) \geq g(t,u_2,v_2)$.
(H3) There exists an function $g_1(t,w) : [0,1]\times(0,+\infty) \to [0,+\infty)$ such that $g_1(t,w)$ is nonincreasing in $u \leq a$ for each fixed $t \in [0,1]$, i.e. if $0 < w_1 \leq w_2$ then $g(t,w_1) \geq g(t,w_2)$
and each fixed $0 < r \leq a$

$$0 < \int_0^1 g_1(s, rs(1-s))ds < \infty.$$

So, we assume additionaly that the function $g(t,u,v)$ satisfies

$$g(t,u,v) \leq g_1(t,u+|v|), \qquad t \in [0,1], \qquad u \in (0,+\infty), \qquad v \in (-\infty,0).$$

Let us introduce the following notations:

$$D_1 = \int_0^1 \int_0^1 K(\tau,\tau)\,K(\tau,s)\,ds\,d\tau,$$

$$D_2 = \int_0^1 \int_0^1 K(\tau,\tau)\,K(\tau,s) f_1(s)\;ds\,d\tau,$$

$$D_3 = \int_0^1 K(\tau,\tau)d\tau,$$

$$D_4 = \int_0^1 K(s,s)\,f_1(s)\,ds,$$

$$D_5 = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)dsd\tau.$$

Lemma 4. Let $(H1),(H2)$, and $(H3)$ hold. Then for all $u \in P \cap \overline{B}_R/B_r$ where $r < a < R$ the following hold

$$(Tu)(t) \le \mu D_3 \|u\|_0 M + D_4 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right), \quad t \in (0,1),$$

and

$$-(Tu)''(t) \le \mu D_3 \|u\|_0 M + D_4 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right), \quad t \in (0,1),$$

where

$$M = \int\limits_0^1 K(v,v)\, g_1(v, rK(v,v))\, dv + \sup_{z\in(0,R]} \sup_{w\in[r,R]} \int\limits_0^1 K(v,v) g(v,z,w)\, dv.$$

Proof. It is easy to see that $D_1 \le D_3$, and $D_2 \le D_4$. Let $u \in P \cap \overline{B}_R/B_r$, then by Lemma 6, $\|u\|_0 \le \|u''\|_0$ and by Corollary 7, $\|u\|_2 = \|u''\|_0$. Thus $r \le \|u''\|_0 \le R$. Also since $u \in P$ we have $-u''(t) \ge K(t,t)\|u''\|_0, \quad u(t) \ge K(t,t)\|u\|_0, \quad t \in [0,1]$.

By Lemma 1. and $(H1) - (H3)$ we have

$$Tu(t) = \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int_0^1 K(s,v)\, g(v,u(v),u''(v))\, dv\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau =$$

$$= \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int\limits_{u(v)+|u''(v)|\le a} K(s,v)\, g(v,u(v),u''(v))\, dv\, ds\, d\tau +$$

$$+\mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int\limits_{u(v)+|u''(v)|\ge a} K(s,v)\, g(v,u(v),u''(v))\, dv\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau) K(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau$$

$$\le \mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\, \|u\|_0 \int\limits_{u(v)+|u''(v)|\le a} K(v,v)\, g_1(v, rK(v,v))\, dv\, ds\, d\tau$$

$$+\mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\, \|u\|_0 \sup_{w\in(0,R]} \sup_{z\in[r,R]} \int\limits_{u(v)+|u''(v)|\ge a} K(v,v) g(v,w,z)\, dv\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s)\, f_1(s)\, ds\, d\tau \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right)$$

$$\le \mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\, \|u\|_0 \int\limits_0^1 K(v,v)\, g_1(v, rK(v,v))\, dv\, ds\, d\tau +$$

$$+\mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s)\, \|u\|_0 \,[\sup_{w\in(0,R]} \sup_{z\in[r,R]} \int\limits_0^1 K(v,v) g(v,w,z)\, dv]\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s)\, f_1(s)\, ds\, d\tau \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right)$$

$$\leq \mu D_1 M \|u\|_0 + D_2 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right)$$

$$\leq \mu D_3 M \|u\|_0 + D_4 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right),$$

and similarly we also have

$$-(Tu)''(t) \leq \mu \int_0^1 \int_0^1 K(\tau,\tau)\, K(\tau,s) ds\, d\tau\, \|u\|_0 M +$$

$$+ \int_0^1 K(s,s)\, f_1(s)\, ds\, d\tau \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right)$$

$$\leq \mu D_3 \|u\|_0 M + D_4 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right).$$

This finishes the proof. □

Lemma 5. $T(P) \subset P$ and $T : P \cap (\overline{B}_R/B_r) \to P$ is completely continuous.

Proof. First, we prove that $T(P) \subset P$. To do this, let $u \in P$, then we define mapping $T : P \to C^2[0,1]$ by (2.6). Then for any $u \in P$, it is clear that

$$(Tu)''(t) = -\mu \int_0^1 K(t,s)u(s) \int_0^1 K(s,v)g(v,u(v),u''(v))dvds -$$

$$- \int_0^1 K(t,s)f(s,u(s),u''(s))ds \leq 0. \tag{2.10}$$

By Lemma 3,

$$Tu(t) \geq K(t,t)\, \|Tu\|_0, \qquad t \in [0,1]$$

and

$$-(Tu)''(t) \geq K(t,t)\, \left\|(Tu)''\right\|_0 \qquad t \in [0,1].$$

Hence $T(P) \subset P$.

Let $V \subset P \bigcap (\overline{B}_R/B_r)$ be a bounded set. Then there exists a $d > 0$, such that $\sup\{\|u\|_2 : u \in V\} = d$.

First we prove $T(V)$ is bounded. Since $\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}$, we have $u(t) + |u''(t)| \leq \|u\|_0 + \|u''\|_0 \leq 2d$, for all $t \in [0,1]$. Let $M_d = \sup\{f_2(w) : w \in [0,2d]\}$. Now from Lemma 4 we have for any $u \in V$ and $t \in [0,1]$ that

$$|Tu(t)| = \left| \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\, u(s) \int_0^1 K(s,v)\, g(v,u(v),u''(v))\, dv\, ds\, d\tau \right.$$

$$\left. + \int_0^1 \int_0^1 K(t,\tau)K(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau \right|$$

$$\leq \mu D_1 \|u\|_0 M + D_2 \sup_{s\in(0,1)} f_2\left(u(s) + \left|u''(s)\right|\right) \leq \mu D_1 d M + M_d D_2. \tag{2.11}$$

We have a similar type inequality for $|(Tu)''(t)|$. Therefore $T(V)$ is bounded.

Next, we prove that $T(V)$ is equicontinuous. Now from Lemma 4 we have for any $u \in V$ and any $t_1, t_2 \in [0,1]$ that

$$|(Tu)(t_1) - (Tu)(t_2)|$$

$$\leq \mu \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\ K(\tau,s)\, u(s) \int_0^1 K(s,v)\, g(v,u(v),u''(v))\, dv\, ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\, K(\tau,s) f(s,u(s),u''(s))\, ds\, d\tau$$

$$\leq \mu \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\ K(\tau,s) ds\, d\tau\, \|u\|_0\, M +$$

$$+ \int_0^1 \int_0^1 |K(t_1,\tau) - K(t_2,\tau)|\, K(\tau,s)\, f_1(s) f_2(u(s) + |u''(s)|)\, ds\, d\tau$$

$$\leq \mu 2\, |t_1 - t_2| \int_0^1 \int_0^1 K(s,s)\, K(s,v) dv\, ds\ \|u\|_0\, M +$$

$$+2M_d\, |t_1 - t_2| \int_0^1 \int_0^1 K(s,s)\, f_1(s)\, ds\, d\tau$$

$$\leq (\mu 2 D_3 d M + M_d D_4)\, |t_1 - t_2|\, .$$

We have a similar type inequality for $|(Tu)''(t_1) - (Tu)''(t_2)|$.

Therefore $T(V)$ is equicontinuous.

Next, we prove that T is continuous. Suppose $u_n, u \in P \cap (\overline{B}_R / B_r)$ and $\|u_n - u\|_2 \to 0$ which implies that $u_n(t) \to u(t), u_n''(t) \to u''(t)$ uniformly on $[0,1]$. Similarly for $f(t,u,v) \leq f_1(t)\, f_2(|u| + |v|)$, $f_2(|u_n(t)| + |u_n''(t)|) \to f_2(|u(t)| + |u''(t)|)$ uniformly on $[0,1]$ and $g(t,u_n(t)) \to g(t,u(t))$ uniformly on $[0,1]$. The assertion follows from the estimate

$$|Tu_n(t) - Tu(t)|$$

$$\leq \mu \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s) \mid u_n(s) \int_0^1 K(s,v) g(v,u_n(v),u_n''(v))\, dv -$$

$$-u(s) \int_0^1 K(s,v) g(v,u(v),u''(v))\, dv \mid\ ds\, d\tau +$$

$$+ \int_0^1 \int_0^1 K(t,\tau)\, K(\tau,s)\ |f_1(s)|\ \mid f_2(u_n(s) + |u_n''(s)|) - f_2(u(s) + |u''(s)|) \mid ds\, d\tau,$$

and the similar estimate for $|(Tu_n)''(t) - (Tu)''(t)|$ by an application of the standard theorem on the convergence of integrals.

The Ascoli-Arzela theorem guarantees that $T : P \to P$ is completely continuous.

This finishes the proof. □

Lemma 6. If $u(0) = u(1) = 0$ and $u \in C^2[0,1]$, then $\|u\|_0 \leq \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.

Proof. Since $u(0) = u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s) ds$, $t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|\, ds \leq \int_0^1 |u''(s)|\, ds \leq \|u''\|_0$, $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$.

Since $u(0) = 0$, we have $u(t) = \int_0^t u'(s)ds$, $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|\, ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$.

This finishes the proof. □

Corollary 7. Let $r > 0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_2 = \|u''\|_0 = r$.

3 Main Results

In this section, we present our main results.

Theorem 1. Let $(H1)$,$(H2)$ and $(H3)$ hold. Assume that the following condition holds
$(H4)$

$$\limsup_{w\to 0^+} \frac{f_2(w)}{w} = 0,$$

and

$$\liminf_{|v|\to\infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \inf_{u\in[0,+\infty)} \frac{f(t,u,v)}{|v|} = \infty.$$

If $\mu \in (0, \frac{1}{4D_1 M_r})$, then problem (1.3) has at least one positive solution.

Proof. We divide the rather long proof into three steps.

(I) Firstly, we will prove that the first part of assumptions (i) of Lemma 2 is satisfied.
To do this, let us choose $0 < c_1 \leq \frac{1}{8D_2}$. Then by $(H4)$, there exist $0 < r < \frac{a}{2}$ such that

$$f_2(u + |v|) \leq c_1\,(u + |v|)\,, \quad 0 \leq u + |v| \leq 2\,r.$$

Let $u \in \partial B_r \cap P$, then by Corollary 7, $\|u\|_2 = \|u''\|_0 = r$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq r$, $|u''(t)| \leq \|u''\|_0 = r$, $\forall t \in [0,1]$. Thus $0 \leq u(t)+|u''(t)| \leq 2\,r$, $\forall t \in [0,1]$. Let

$$M_r = \int\limits_0^1 K(v,v)\, g_1(v, rK(v,v))\, dv.$$

Thus, by Lemma 4, and $(H1 - H3)$ we have

$$(Tu)(t) \leq \mu \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)u(s)dsd\tau \int\limits_0^1 K(v,v)\, g_1(v, rK(v,v))\, dv +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)f_2(u(s) + \left|u''(s)\right|)dsd\tau$$

$$\leq \mu \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)dsd\tau\, \|u\|_0\, M_r + \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)c_1(u(s) + \left|u''(s)\right|)dsd\tau$$

$$\leq \mu D_1 \|u\|_0\, M_r + c_1 D_2(\|u\|_0 + \left\|u''\right\|_0) \leq \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2$$

$$\leq \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 \leq \frac{1}{2}\|u\|_2\,, \qquad \forall u \in \partial B_r \cap P, \quad t \in [0,1].$$

Consequently,

$$\|Tu\|_0 \leq \frac{1}{2}\|u\|_2, \qquad \forall u \in \partial B_r \cap P. \tag{3.1}$$

Similarly we also have

$$(Tu)''(t) = -\mu \int_0^1 K(t,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))dvds-$$
$$-\int_0^1 K(t,s)f(t,u(s),u''(s))ds.$$

Hence

$$\left|(Tu)''(t)\right| \leq \mu \int_0^1 K(s,s)ds\,\|u\|_0 M_r + \int_0^1 K(s,s)f_1(s)f_2(|u| + \left|u''\right|)ds$$
$$\leq \mu D_3 \|u\|_0 M_r + c_1 D_4(\|u\|_0 + \left\|u''\right\|_0) \leq \mu D_1 \|u\|_0 M_r + c_1 D_2(\|u\|_0 + \left\|u''\right\|_0)$$
$$\leq \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2 \leq \frac{1}{2}\|u\|_2, \quad \forall u \in \partial B_r \cap P, \quad t \in [0,1].$$

Consequently,

$$\left\|(Tu)''\right\|_0 \leq \frac{1}{2}\|u\|_2, \qquad \forall u \in \partial B_r \cap P. \tag{3.2}$$

Using (3.1) and (3.2) we have

$$\|Tu\|_2 \leq \|Tu\|_0 + \left\|(Tu)''\right\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_r \cap P. \tag{3.3}$$

Thus, we have $\|Tu\|_2 \leq \|u\|_2$, for all $u \in \partial B_r \cap P$. This proves one of assumptions appearing Lemma 2.

(II) Secondly, we will prove that the second part of assumptions (i) of Lemma 2 is satisfied. To do this, let us choose $c_2 \geq \frac{1}{\sigma D_5}$. Then by condition $(H4)$, there exists $R_1 > 0$ such that

$$f(t,u,v) \geq c_2 |v|, \quad \forall u \in R^+, \forall |v| \geq R_1, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let $R > \max\{\frac{R_1}{\sigma}, a\}$. Let $u \in \partial B_R \cap P$, i.e. $\|u''\|_0 = R$. Thus by using (2.7) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left|u''(t)\right| \geq \sigma \left\|u''\right\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 1, we have

$$(Tu)(\frac{1}{2}) \geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)u(s)K(s,v)g(v,u(v),u''(v))dvdsd\tau+$$
$$+\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)f(t,u(s),u''(s))dsd\tau$$

$$\geq c_2 \int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)\left|u''(s)\right| dsd\tau \geq c_2\sigma \int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)dsd\tau \left\|u''\right\|_0$$
$$\geq \left\|u''\right\|_0$$

so

$$(Tu)(\frac{1}{2}) \geq \|u''\|_0 = \|u\|_2\,, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \quad \|Tu\|_2\,, \quad \forall u \in \partial B_R \cap P. \tag{3.4}$$

Thus, we have $\|u\|_2 \leq \|Tu\|_2$, for all $u \in \partial B_R \cap P$. This proves one of assumptions appearing Lemma 2.

(III) Finally, we will prove that $T : P \cap (\overline{B}_R \backslash B_r) \to P$ is a completely continuous operator. By Lemma 5, the Ascoli-Arzela theorem guarantees that $T : P \cap (\overline{B}_R \backslash B_r) \to P$ is a completely continuous.

Then due to Lemma 2, by (3.3) and (3.4) inequality we see that the problem (1.3) has at least one positive solution.

This finishes the proof. □

Theorem 2. Let $(H1)$,$(H2)$ and $(H3)$ hold. Assume that the following conditions hold

$(H5)$

$$\liminf_{|u|+|v|\to 0^+} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,u,v)}{|u|+|v|} = \infty,$$

and

$$\liminf_{|v|\to\infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \inf_{u\in[0,+\infty)} \frac{f(t,u,v)}{|v|} = \infty.$$

$(H6)$ there exists $0 < \varrho < \frac{a}{2}$ such that

$$\sup_{w\in[0,a]} f_2(w) \leq \frac{\varrho}{4D_4}. \tag{3.5}$$

If $\mu \in (0, \frac{1}{4D_3 M})$, then problem (1.3) has at least two positive solutions.
We note for the argument below that $D_1 \leq D_3$ and $D_2 \leq D_4$.

Proof. We divide the rather long proof into four steps.

(I) Firstly, we will prove that the first part of assumptions (i) of Lemma 2 is satisfied.
To do this, by condition $(H6)$ there exists $0 < \varrho < \frac{a}{2}$ such that (3.5) is fulfilled. Let $u \in \partial B_\varrho \cap P$, by Corollary 7, $\|u''\|_0 = \varrho$, $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq \varrho$, $|u''(t)| \leq \|u''\|_0 = \varrho$, $\forall t \in [0,1]$. Thus $0 \leq |u(t)| + |u''(t)| < 2\varrho$, $\forall t \in [0,1]$. Let

$$M_\varrho = \int_0^1 K(v,v)\, g(v, \varrho K(v,v))\, dv.$$

By condition $(H6)$, $\forall u \in \partial B_\varrho \cap P$ and $t \in [0,1]$, we have

$$(Tu)(t) \leq \mu \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)dsd\tau \, \|u\|_0 \, M_\varrho +$$

$$+ \int_0^1 \int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)f_2(u(s) + |u''(s)|)dsd\tau$$

$$\leq \mu D_1 \|u\|_0 M_\varrho + \frac{\varrho}{4D_4}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)ds$$

$$\leq \frac{1}{4}\|u\|_0 + \frac{1}{4}\varrho = \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u''\|_0 = \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2$$

$$\leq \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 \leq \frac{1}{2}\|u\|_2, \quad \forall u \in \partial B_\varrho \cap P, \quad t \in [0,1].$$

Consequently, we get

$$\|Tu\|_0 \leq \|u\|_2, \qquad \forall u \in \partial B_\varrho \cap P. \tag{3.6}$$

Similarly we also have

$$(Tu)''(t) = -\mu \int_0^1 K(t,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))dvdsd\tau -$$

$$-\int_0^1 K(t,s)f(s,u(s),u''(s))dsd\tau.$$

Hence

$$\left|(Tu)''(t)\right| \leq \mu \int_0^1 K(s,s)ds\,\|u\|_0 M_\varrho + \int_0^1 K(s,s)f_1(s)f_2(u(s)+\left|u''(s)\right|)ds$$

$$\leq \mu D_3 \|u\|_0 M_\varrho + \frac{\varrho}{4D_4}\int_0^1 K(s,s)f_1(s)ds \leq \frac{1}{4}\|u\|_0 + \frac{1}{4}\varrho$$

$$= \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2 \leq \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 \leq \frac{1}{2}\|u\|_2,$$

$$\forall u \in \partial B_\varrho \cap P, \quad t \in [0,1].$$

Consequently,

$$\left\|(Tu)''\right\|_0 \leq \frac{1}{2}\|u\|_2, \qquad \forall u \in \partial B_\rho \cap P. \tag{3.7}$$

Using (3.6) and (3.7) we have

$$\|Tu\|_2 \leq \|Tu\|_0 + \left\|(Tu)''\right\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_\rho \cap P. \tag{3.8}$$

Thus, we have $\|Tu\|_2 \leq \|u\|_2$, for all $u \in \partial B_\varrho \cap P$. This proves one of assumptions appearing Lemma 2.

(II) Secondly, we will prove that the first part of assumptions (i) of Lemma 2 is satisfied.
To do this, let us choose $c_3 \geq \frac{1}{\sigma D_5}$. The by condition $(H5)$, there exists $0 < r < \varrho$ such that

$$f(t,u,v) \geq c_3\left(u + |v|\right), \quad \forall u \in [0,r], \ \forall |v| \in [0,r], \quad t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Let $u \in \partial B_r \cap P$, by Corollary 7, $\|u''\|_0 = r$, $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have

$$0 \leq u(t) \leq \|u\|_0 \leq r, \quad \left|u''(t)\right| \leq \left\|u''\right\|_0 = \|u\|_2 = r, \quad \forall u \in \partial B_r \cap P.$$

Thus by using (2.7) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left|u''(t)\right| \geq \sigma \left\|u''\right\|_0 = \sigma r, \quad \forall u \in \partial B_r \cap P.$$

The estimate for $(Tu)(\frac{1}{2})$ is similar to that in the proof of Theorem 1 i.e. from Lemma 1 and $(H5)$ we have

$$(Tu)(\frac{1}{2})$$

$$\geq c_3 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)(u(s)+\left|u''(s)\right|)dsd\tau \geq c_3 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)\left|u''(s)\right| dsd\tau$$

$$\geq c_3\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)dsd\tau \left\|u''\right\|_0 \geq \left\|u''\right\|_0 .$$

Thus

$$(Tu)(\frac{1}{2}) \geq \left\|u''\right\|_0 = \left\|u\right\|_2, \quad \forall u \in \partial B_r \cap P.$$

Consequently,

$$\left\|u\right\|_2 \leq \left\|Tu\right\|_0 \leq \left\|Tu\right\|_2, \qquad \forall u \in \partial B_r \cap P. \tag{3.9}$$

Thus, we have $\left\|u\right\|_2 \leq \left\|Tu\right\|_2$, for all $u \in \partial B_r \cap P$. This proves one of assumptions appearing Lemma 2.

(III) Thirdly, we will prove that the first part of assumptions (ii) of Lemma 2 is satisfied.
To do this, we show that for sufficiently large $R > a$, it holds

$$\left\|Tu\right\|_2 \geq \left\|u\right\|_2, \qquad \forall u \in \partial B_R \cap P.$$

To see this we choose $c_2 \geq \frac{1}{\sigma D_5}$. Due to condition $(H5)$, there exist $R_1 > 0$ such that

$$f(t,u,v) \geq c_2 \left|v\right|, \quad \forall u \in R^+, \quad \forall \left|v\right| \geq R_1, \quad t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Let $R > \max\{\frac{R_1}{\sigma}, a\}$. Let $u \in \partial B_R \cap P$, by Corollary 7, $\left\|u''\right\|_0 = R$. Thus by using (2.7) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left|u''(t)\right| \geq \sigma \left\|u''\right\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 1, $(H1)$ and $(H4)$, we have

$$(Tu)(\frac{1}{2}) \geq c_2 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)\left|u''(s)\right| dsd\tau$$

$$\geq c_2\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K(\frac{1}{2},\tau)K(\tau,s)dsd\tau \left\|u''\right\|_0 \geq \left\|u''\right\|_0$$

so

$$(Tu)(\frac{1}{2}) \geq \left\|u''\right\|_0 = \left\|u\right\|_2, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\left\|u\right\|_2 \leq \left\|Tu\right\|_0 \leq \left\|Tu\right\|_2, \quad \forall u \in \partial B_R \cap P. \tag{3.10}$$

Thus, we have $\|u\|_2 \leq \|Tu\|_2$, for all $u \in \partial B_R \cap P$. This proves one of assumptions appearing Lemma 2.

(IV) Finally, we will prove that the problem (1.3) has at least two positive solutions. We remind that $r < \rho < a < R$, and by Lemma 5, the Ascoli-Arzela theorem guarantees that $T : P \cap (\overline{B}_R \backslash B_\rho) \to P$ and $T : P \cap (\overline{B}_\rho \backslash B_r) \to P$ are completely continuous operators.

Then due to Lemma 2, by (3.8) and (3.9) inequality we see that the problem (1.3) has at least one positive solution in $\left(\overline{B}_\rho \backslash B_r\right) \cap P$. Similarly, by (3.8) and (3.10) inequality we see that the problem (1.3) has at least one positive solution in $\left(\overline{B}_R \backslash B_\rho\right) \cap P$. Then, we know that T has at least two fixed points in $\left(\overline{B}_R \backslash B_\rho\right) \cap P$ and $\left(\overline{B}_\rho \backslash B_r\right) \cap P$, i.e. problem (1.3) has at least two positive solutions.

This finishes the proof. □

4 Examples

In this section, we will give two examples to illustrate Theorem 1 and Theorem 2.

Example 1. Consider the following boundary problem

$$\begin{aligned} u^{(4)} &= \varphi u + \frac{(u + |u''|)^2}{t(1-t)}, \quad 0 < t < 1 \\ -\varphi'' &= \mu\left(\frac{1}{\sqrt{u + |u''|}} + (u + |u''|)^{\frac{1}{3}}\right), \ 0 < t < 1 \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \\ \varphi(0) &= \varphi(1) = 0. \end{aligned} \tag{4.1}$$

We can choose $f_1(t) = \frac{1}{t(1-t)}$, $f_2(u + |u''|) = (u + |u''|)^2$, $g_1(t, u, |u''|) = \frac{1}{\sqrt{u+|u''|}} + (u + |u''|)^{\frac{1}{3}}$. Obviously, $(H_1), (H_2), (H_3)$ and (H_4) are satisfied. Therefore, by Theorem 1, the problem (4.1) has at least one positive solution when $\mu \in (0, \frac{1}{4D_1 M_r})$.

Example 2. Consider the following boundary problem

$$\begin{aligned} u^{(4)} &= \varphi u + \frac{(u + |u''|)^2 + (u + |u''|)^{\frac{1}{2}}}{t(1-t)}, \quad 0 < t < 1 \\ -\varphi'' &= \mu\left(\frac{1}{\sqrt{u + |u''|}} + (u + |u''|)^{\frac{1}{4}}\right), \ 0 < t < 1 \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \\ \varphi(0) &= \varphi(1) = 0. \end{aligned} \tag{4.2}$$

We can choose $f_1(t) = \frac{1}{t(1-t)}$, $f_2(u+|u''|) = (u+|u''|)^2+(u+|u''|)^{\frac{1}{2}}$, $g_1(t, u, |u''|) = \frac{1}{\sqrt{u+|u''|}}+(u+|u''|)^{\frac{1}{3}}$. Obviously, $(H_1), (H_2), (H_3)$ and (H_5) are satisfied. Therefore, by Theorem 2, the problem (4.2) has at least two positive solutions when $\mu \in (0, \frac{1}{4D_3 M})$.

5 Conclusions

This paper investigates the existence of positive solutions for a nonlinear fourth-order differential system using a fixed point theorem of cone expansion and compression type of norm type. The nonlinear terms may be singular with respect to the time and space variables. The problem comes from the deformation analysis of an elastic beam in the equilibrium state, whose two ends are simply supported. The results obtained herein generalize and improve some known results including singular and non-singular cases.

Competing Interests

Author has declared that no competing interests exist.

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