On Continuity of Grill Topological Spaces VIA Regular Generalized $G_\omega$–Closed Sets

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Author’s contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2022/v18i1130427

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc. are available here:  https://www.sdiarticle5.com/review-history/88413

Received: 29 June 2022
Accepted: 01 September 2022
Published: 09 September 2022

Abstract

Aims/Objectives: In this paper, we study the continuity property in grill topological spaces via generalized $G_\omega$–closed sets and regular generalized $G_\omega$–closed sets. These notions are generalized $G_\omega$–continuous functions which is weak form of $g_\omega$–continuous functions, generalized $G_\omega$–irresolute functions, regular generalized $G_\omega$–continuous functions which is weak form of $rg_\omega$–continuous functions, regular generalized $G_\omega$–irresolute functions and investigates some of its properties in grill topological spaces.

Keywords: Grill topological spaces; continuous functions.

2010 Mathematics Subject Classification: Primary 54C08, 54C05

1 Introduction


In this paper, we introduce and investigate new notions of generalized \( G^- \)-closed sets and regular generalized \( G^- \)-closed sets in grill topological spaces. These notions are called generalized \( G^- \)-continuous functions and regular generalized \( G^- \)-continuous functions in grill topological spaces. In Section 3, we introduce and investigate the notions of generalized \( G^- \)-closed sets called generalized \( G^- \)-continous functions, generalized \( G^- \)-closed functions, generalized \( G^- \)-irresolute functions and introduce investigate the notion of \( G^\omega^- \)-irresolute functions. In Section 4, we introduce and investigate new the notions of generalized \( G^- \)-continuous functions, generalized \( G^- \)-closed functions and generalized \( G^- \)-irresolute functions called regular generalized \( G^- \)-continuous functions, regular generalized \( G^\varphi^- \)-closed functions and regular generalized \( G^- \)-irresolute functions, respectively.

2 Preliminaries

**Definition 2.1.** A function \( f : (X, \tau) \rightarrow (Y, \rho) \) of a topological space \((X, \tau)\) into a topological space \((Y, \rho)\) is called:

1. \( g^- \)-continuous function [12] if \( f^{-1}(F) \) is \( g^-\)-closed set in \((X, \tau)\) for every closed set \( F \) of \((Y, \rho)\);
2. \( \omega^- \)-continuous function [1] if \( f^{-1}(V) \) is \( \omega^- \)-open set in \((X, \tau)\) for every open set \( V \) of \((Y, \rho)\);
3. \( g\omega^- \)-continuous function [4] if \( f^{-1}(F) \) is \( g\omega^-\)-closed set in \((X, \tau)\) for every closed set \( F \) of \((Y, \rho)\);
4. \( g\omega^- \)-closed function [4] if the image of every closed set of \((X, \tau)\) is \( g\omega^-\)-closed set in \((Y, \rho)\);
5. \( rg\omega^- \)-continuous function [5] if \( f^{-1}(F) \) is \( rg\omega^-\)-closed set in \((X, \tau)\) for every \( \omega^- \)-closed set \( F \) of \((Y, \tau)\).

By \( Cl(A) \) and \( Int(A) \), we mean the closure set and the interior set of \( A \) in topological space \((X, \tau)\), respectively.

A collection \( \mathcal{G} \) of subsets of a topological space \((X, \tau)\) is said to be a grill [13] on \( X \) if \( \mathcal{G} \) satisfies the following conditions:

1. \( \emptyset \notin \mathcal{G} \);
2. \( A \in \mathcal{G} \) and \( A \subseteq B \) implies that \( B \in \mathcal{G} \);
3. \( A, B \subseteq X \) and \( A \cup B \in \mathcal{G} \) implies that \( A \in \mathcal{G} \) or \( B \in \mathcal{G} \).

For a grill \( \mathcal{G} \) on a topological space \( X \), an operator from the power set \( P(X) \) of \( X \) to \( P(X) \) was defined in [14] the following manner: For any \( A \in P(X) \),

\[ \Phi(A) = \{ x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood} U \text{ of } x \}. \]

Then the operator \( \Psi : P(X) \rightarrow P(X) \), given by \( \Psi(A) = A \cup \Phi(A) \), for \( A \in P(X) \), was also shown in [14] to be a Kuratowski closure operator, defining a unique topology \( \tau_\mathcal{G} \) on \( X \) such that \( \tau \subseteq \tau_\mathcal{G} \).

This topology defined by:

\[ \tau_\mathcal{G} = \{ U \subseteq X : \Psi(X - U) = X - U \}, \]
where \( \tau \subseteq \tau_G \) and for any \( A \subseteq X \), \( \Psi(A) = gCl(A) \) such that \( gCl(A) \) denotes the set of all closure points of \( A \) in a topological space \( (X, \tau_G) \). The set of all interior points of \( A \) in a topological space \( (X, \tau_G) \) denoted by \( gInt(A) \).

**Definition 2.2.** A function \( f : (X, \tau, \mathcal{G}) \to (Y, \rho) \) of a grill topological space \( (X, \tau, \mathcal{G}) \) into a topological space \( (Y, \rho) \) is called:

1. \( \mathcal{G}^\omega \)-continuous function \([9]\) if \( f^{-1}(U) \) is \( \mathcal{G}^\omega \)-open set in \( (X, \tau, \mathcal{G}) \) for every open set \( U \) in \( (Y, \rho) \);
2. \( \mathcal{G}^\omega \)-closed function \([9]\) if \( f(F) \) is a closed set in \( (Y, \rho) \) for every \( \mathcal{G}^\omega \)-closed set \( F \) in \( (X, \tau, \mathcal{G}) \);
3. \( \mathcal{G}^\omega \)-continuous function \([10]\) if \( f^{-1}(U) \) is \( \mathcal{G}^\omega \)-open set in \( (X, \tau, \mathcal{G}) \) for every open set \( U \) in \( (Y, \rho) \);
4. \( \mathcal{G}^\omega \)-closed function \([10]\) if \( f(F) \) is a closed set in \( Y \) for every \( \mathcal{G}^\omega \)-closed set \( F \) in \( (X, \tau, \mathcal{G}) \).

**Definition 2.3.** A subset \( A \) of a topological space \( (X, \tau) \) is called:

1. \( \alpha \)-open set \([15]\) if \( A \subseteq Int(Cl(Int(A))) \). The complement of \( \alpha \)-open set is called \( \alpha \)-closed set;
2. \( \omega \)-open set \([16]\) if for each \( x \in A \), there is an open set \( U_x \in \tau \) containing \( x \) such that \( U_x - A \) is a countable set. The complement of \( \omega \)-open set is called \( \omega \)-closed set;
3. \( \alpha - \omega \)-open set \([6]\) if \( A \subseteq Int_\omega(Cl(Int_\omega(A))) \). The complement of \( \alpha - \omega \)-open set is called \( \alpha - \omega \)-closed set.

**Definition 2.4.** A subset \( A \) of a grill topological space \( (X, \tau, \mathcal{G}) \) is called:

1. \( \mathcal{G} \) - \( \alpha \)-open set \([7]\) if \( A \subseteq Int(\Psi(Int(A))) \). The complement of \( \mathcal{G} - \alpha \)-open set is called \( \mathcal{G} - \alpha \)-closed set;
2. \( \mathcal{G}^\omega \)-open set \([8]\) if \( A \subseteq Cl(\Psi(Int_\omega(A))) \). The complement of \( \mathcal{G}^\omega \)-open set is called \( \mathcal{G}^\omega \)-closed set;
3. \( \mathcal{G}^\omega \)-open set \([17]\) if \( A \subseteq Int(\Psi(Int_\omega(A))) \). The complement of \( \mathcal{G}^\omega \)-open set is called \( \mathcal{G}^\omega \)-closed set. The set of all \( \mathcal{G}^\omega \)-open sets in \( X \) is denoted by \( G^\omega_0o(X, \tau) \) and the set of all \( \mathcal{G}^\omega \)-closed sets in \( X \) is denoted by \( G^\omega_0c(X, \tau) \);
4. \( \mathcal{G}^\omega \)-closed set \([8]\) if \( gCl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open set of \( (X, \tau, \mathcal{G}) \). The complement of \( \mathcal{G}^\omega \)-closed set is called \( \mathcal{G}^\omega \)-open set;
5. \( \mathcal{G}_{rg} \)-closed set \([11]\) if \( gCl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular-open set of \( (X, \tau, \mathcal{G}) \). The complement of \( \mathcal{G}_{rg} \)-closed set is called \( \mathcal{G}_{rg} \)-open set.

From the definition stated above we obtain the following figure of implications:

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continuous function

\( \omega \)-continuous function \( \xrightarrow{g\omega \text{-continuous function}} \) \( \Gamma_{g\omega} \)-continuous function

\( \mathcal{G}^\omega \)-continuous function
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**Fig. 1.**
3 \(G^\omega_g\)-Continuous Functions

Definition 3.1. A function \(f : (X, \tau, G) \to (Y, \rho)\) of a grill topological space \((X, \tau, G)\) into a space \((Y, \rho)\) is called \(G^\omega_g\)-continuous function if \(f^{-1}(U)\) is \(G^\omega_g\)-closed set in \((X, \tau, G)\) for every closed set \(U\) in \((Y, \rho)\).

Example 3.1. Any function \(f : (X, \tau, G) \to (Y, \rho)\) of any countable grill topological space \((X, \tau, G)\) into any space \((Y, \rho)\) is \(G^\omega_g\)-continuous function.

Definition 3.2. A function \(f : (X, \tau, G) \to (Y, \tau, H)\) of a grill topological space \((X, \tau, G)\) into a grill topological space \((Y, \tau, H)\) is called:

1. \(G^\omega_g\)-irresolute function if \(f^{-1}(U)\) is \(G^\omega_g\)-closed set in \((X, \tau, G)\) for every \(G^\omega_g\)-closed set in \((Y, \tau, H)\).
2. \(G^\omega_g\)-continuous function if \(f^{-1}(U)\) is \(G^\omega_g\)-closed set in \((X, \tau, G)\) for every \(G^\omega_g\)-closed set in \((Y, \tau, H)\).

Theorem 3.2. Let \(f : (X, \tau, G) \to (Y, \rho)\) be a function of a grill topological space \((X, \tau, G)\) into a space \((Y, \rho)\). Then \(f\) is \(G^\omega_g\)-continuous function if \(f^{-1}(U)\) is \(G^\omega_g\)-open set in \((X, \tau, G)\) for every open set \(U\) in \((Y, \rho)\).

Proof. Let \(f\) be \(G^\omega_g\)-continuous function. Let \(U\) be open set in \((Y, \rho)\) then \(Y - F\) is closed set in \((Y, \rho)\). Since \(f\) is \(G^\omega_g\)-continuous function. Then \(f^{-1}(Y - F) = X - f^{-1}(U)\) is \(G^\omega_g\)-closed set in a grill topological space \((X, \tau, G)\). Therefore \(f^{-1}(U)\) is \(G^\omega_g\)-open set in \((X, \tau, G)\). Hence \(f\) is \(G^\omega_g\)-continuous function.

It is clear that every \(G^\omega_g\)-continuous function is \(G^\omega_g\)-continuous function but the converse of this fact need not be true.

Example 3.3. Let \(f : (\mathbb{R}, \tau, G) \to (Y, \rho)\) be a function defined by:
\[
f(x) = \begin{cases} 
1, & x \in (0, \frac{1}{2}] \\
2, & x \in \mathbb{R} - (0, \frac{1}{2}] 
\end{cases}
\]
where \(Y = \{1, 2\}\),
\[
\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \ G = P(\mathbb{R}) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1\}\}.
\]

The function \(f\) is \(G^\omega_g\)-continuous, since \(f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]\) and \(f^{-1}(Y) = \mathbb{R}\) are \(G^\omega_g\)-closed sets in \((\mathbb{R}, \tau, G)\). The function \(f\) is not \(G^\omega_g\)-continuous function, since \(f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2}]\) is not \(G^\omega_g\)-closed set.

It is clear that every \(g\omega\)-continuous function is \(G^\omega_g\)-continuous function but the converse of this fact need not be true.

Example 3.4. Let \(f : (X, \tau, G) \to (Y, \rho)\) be a function defined by:
\[
f(x) = \begin{cases} 
1, & x \in X - B \\
2, & x \in B 
\end{cases}
\]
where \(X, B\) and \(A\) are uncountable sets, \(B \subset A\) and \(Y = \{1, 2\}\),
\[
\tau = \{\emptyset, X, A\}, \ G = P(X) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{1\}\}.
\]

Then the function \(f\) is \(G^\omega_g\)-continuous but \(f\) is not \(g\omega\)-continuous. Since \(f^{-1}(\{2\}) = B\) and \(f^{-1}(Y) = X\) are \(G^\omega_g\)-closed sets in \((X, \tau, G)\) but \(f^{-1}(\{2\}) = B\) is not \(g\omega\)-closed set.
It is clear that every $\mathcal{G}_c^\omega$-continuous function is $\mathcal{G}_g^\omega$-continuous function but the converse of this fact need not be true.

**Example 3.5.** In Example (3.3), we note that $f$ is $\mathcal{G}_c^\omega$-continuous function but the function $f$ is not $\mathcal{G}_c^\omega$-continuous function. Since $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2})$ is not $\mathcal{G}_c^\omega$-closed set in $(X, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}^\omega$-irresolute function is $\mathcal{G}_g^\omega$-continuous function but the converse of this fact need not be true.

**Example 3.6.** Let $f : (\mathbb{R}, \tau, \mathcal{G}) \to (\mathbb{R}, \rho, \mathcal{H})$ be the identity function where
\[
\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \quad \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \rho = \{\emptyset, \mathbb{R}\}.
\]

The function $f$ is $\mathcal{G}^\omega$-continuous function since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are $\mathcal{G}^\omega$-closed sets in $(\mathbb{R}, \tau, \mathcal{G})$ but $f$ is $\mathcal{G}^\omega$-irresolute function. Since $\mathbb{R} - (0, 1)$ is $\mathcal{G}^\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{H})$ but it is not $\mathcal{G}_g^\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}_c^\omega$-irresolute function is $\mathcal{G}_g^\omega$-continuous function but the converse of this fact need not be true.

**Example 3.7.** In Example (3.6), we note that $f$ is $\mathcal{G}_c^\omega$-continuous function, since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are $\mathcal{G}_c^\omega$-closed sets in $(\mathbb{R}, \tau, \mathcal{G})$ but the function $f$ is not $\mathcal{G}_c^\omega$-irresolute function, since $\mathbb{R} - (0, 1)$ is $\mathcal{G}_c^\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{H})$ but it is not $\mathcal{G}_c^\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}^\omega$-irresolute function is $\mathcal{G}_c^\omega$-irresolute function but the converse of this fact need not be true.

**Example 3.8.** In Example (3.3), if $f : (\mathbb{R}, \tau, \mathcal{G}) \to (\mathbb{R}, \rho, \mathcal{H})$, we note that $f$ is $\mathcal{G}^\omega$-irresolute function, but the function $f$ is not $\mathcal{G}^\omega$-irresolute function, since $\mathbb{R} - (0, \frac{1}{2})$ is $\mathcal{G}^\omega$-closed set in $(Y, \tau, \mathcal{H})$ but $f^{-1}(\{2\}) = \mathbb{R} - (0, \frac{1}{2})$ is not $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$.

The following figure is an enlargement of fig. 1. and an theorems above we have the following figure:

![Diagram](image_url)

**Theorem 3.9.** If $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is $\mathcal{G}_c^\omega$-continuous, then for each $x \in X$ and each open set $V$ in $(Y, \rho)$ with $f(x) \in V$, there exists $\mathcal{G}_g^\omega$-open set $U$ in $(X, \tau, \mathcal{G})$ such that $x \in U$ and $f(U) \subseteq V$. 

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Proof. Let $f$ is $G^*_g$-continuous function. Let $x \in X$ and $V$ be any open set in $(Y, \rho)$ containing $f(x)$. Put $U = f^{-1}(V)$. Since $f$ is $G^*_g$-continuous then $U$ is $G^*_g$-open set in $(X, \tau, G)$ such that $x \in U$ and $f(U) \subseteq V$.

The converse of the above Theorem is not true in general as the following Example shows.

Example 3.10. In Example (3.4), if

$$\tau = \{\emptyset, X, B\}, \quad G = P(X) - \emptyset \quad \text{and} \quad \rho = \{\emptyset, Y, \{1\}\}.$$ 

We note that $f$ is satisfies the property stated in Theorem(3.9). The function $f$ is not $G^*_g$-continuous function. Since $\{2\}$ is a closed set in $Y$ but $f^{-1}(\{2\}) = B$ is not $G^*_g$-closed set in $(X, \tau, G)$.

Theorem 3.11. Let $f : (X, \tau, G) \to (Y, \rho)$ be a $G^*_g$-continuous function and let $A$ be a closed subset of $(X, \tau, G)$. Then, the restriction $f|A : (A, \tau_A) \to (Y, \rho)$ is $G^*_g$-continuous.

Proof. Let $F$ be a closed subset of $(Y, \rho)$. Then $(f|A)^{-1}(F) = f^{-1}(F) \cap A$. Since $f$ is $G^*_g$-Continuous, $f^{-1}(F) \in G^*_g C(X, \tau, G)$ and since it is clear that if $A \in G^*_g C(X, \tau, G)$ and $B$ is closed set in $(X, \tau, G)$ then $A \cap B \in G^*_g C(X, \tau, G)$. Therefore, since it is clear that if $A \in G^*_g C(X, \tau, G)$, then $A \in G^*_g C(Y, \tau_Y), (f|A)^{-1}(F) \in G^*_g C(Y, \tau_Y)$. 

Theorem 3.12. For a function $f : (X, \tau, G) \to (Y, \rho)$, the following holds:

1. $f$ is $G^*_g$-continuous function.
2. For each $x \in X$ and each open set $V$ in $(Y, \rho)$ with $f(x) \in V$, there exists $G^*_g$-open set $U$ in $(X, \tau, G)$ such that $x \in U$ and $f(U) \subseteq V$.

Definition 3.3. A function $f : (X, \tau, G) \to (Y, \rho)$ of a grill topological space $(X, \tau, G)$ into a space $(Y, \rho)$ is called:

1. $G^*_g$-closed function if $f(G)$ is a closed set in $(Y, \rho)$ for every $G^*_g$-closed set $G$ in $(X, \tau, G)$.
2. $G^*_g$-open function if $f(G)$ is a open set in $(Y, \rho)$ for every $G^*_g$-open set $G$ in $(X, \tau, G)$.

Example 3.13. $f : (\mathbb{R}, \tau, G) \to (\mathbb{R}, \rho_D)$ be a any function, with

$$\tau = \{\emptyset, \mathbb{R}, \{1, 3\}\} \quad \text{and} \quad G = P(\mathbb{R}) - \emptyset.$$ 

Since $\{1, 3\}$ is a countable set then by Remark(??) any subset of $X$ is a both $G^*_g$-open set and $G^*_g$-closed set. Therefore $f$ is a both $G^*_g$-closed function and $G^*_g$-open function.

Remark 3.1. Let $(X, \tau_I, G)$ be a grill topological space with any set $X$ then any subset of $X$ is a both $G^*_g$-open set and $G^*_g$-closed set.

Theorem 3.14. Let $f : (X, \tau_I, G) \to (Y, \rho)$ be any function then $f$ is $G^*_g$-continuous function.

Proof. Let $A \subseteq Y$ then $f^{-1}(A)$ is a both $G^*_g$-open set and $G^*_g$-closed. Therefore $f$ is $G^*_g$-continuous function.

Theorem 3.15. Every $G^*_g$-closed function is $G^*_g$-closed function.

The proof is obvious.

The converse above Theorem need not be true.
Example 3.16. Let \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho) \) be a function defined by:
\[
f(x) = \begin{cases} 
  b, & x \in \{2, 3\} \\
  a, & x \in \{1\}
\end{cases}
\]
where \( X = \{1, 2, 3\}, Y = \{a, b\}, \tau = \{\emptyset, X, \{1\}\}, \mathcal{G} = \mathcal{P}(X) - \{\emptyset\} \) and \( \rho = \{\emptyset, Y, \{a\}\} \).

We note that \( f \) is \( \mathcal{G}_\tau^G \)-closed function but it is not \( \mathcal{G}_\rho^G \)-closed function. Since \( \{1\} \) is \( \mathcal{G}_\tau^G \)-closed set in \( (X, \tau, \mathcal{G}) \) but \( f(\{1\}) = \{a\} \) it is not closed set in \( (Y, \rho) \).

Theorem 3.17. Every \( \mathcal{G}_\tau^G \)-closed function is \( \mathcal{G}_\rho^G \)-closed function.

The proof is obvious.

The converse above Theorem need not be true.

Example 3.18. Let \( f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho) \) be a function defined by:
\[
f(x) = \begin{cases} 
  1, & x \in B, B \subset (0, 1) \\
  3, & x \in \mathbb{R} - B, x \in (0, 1) \\
  2, & x \in \mathbb{R} - C
\end{cases}
\]
where \( (0, 1) \subset C, Y = \{1, 2, 3\}, \tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \mathcal{G} = \mathcal{P}(\mathbb{R}) - \{\emptyset\} \) and \( \rho = \{\emptyset, Y, \{1, 3\}, \{2, 3\}, \{3\}\} \).

We note that \( f \) is \( \mathcal{G}_\rho^G \)-closed function but it is not \( \mathcal{G}_\tau^G \)-closed function. Since \( \mathbb{R} - B \) is \( \mathcal{G}_\rho^G \)-closed set in \( (\mathbb{R}, \tau, \mathcal{G}) \) but \( f(\mathbb{R} - B) = \{3\} \) it is not closed set in \( (Y, \rho) \).

Theorem 3.19. Let \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho) \) and \( g : (Y, \rho, \mathcal{H}) \rightarrow (Z, \gamma) \) two functions. Then the following hold:
1. \( g \circ f \) is \( \mathcal{G}_\rho^G \)-continuous function if \( g \) is continuous function and \( f \) is \( \mathcal{G}_\tau^G \)-continuous function.
2. \( g \circ f \) is \( \mathcal{G}_\rho^G \)-continuous function if \( g \) is \( \mathcal{G}_\gamma^H \)-continuous function and \( (Y, \rho) \) is \( T_{1/2} \)-space and \( f \) is \( \mathcal{G}_\tau^G \)-continuous function.

Theorem 3.20. Let \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{I}) \) and \( g : (Y, \rho, \mathcal{H}) \rightarrow (Z, \gamma, \mathcal{K}) \) two functions. Then the following hold:
1. \( g \circ f \) is \( \mathcal{G}_\rho^G \)-continuous function if \( g \) is \( \mathcal{G}_\gamma^H \)-continuous function and \( f \) is \( \mathcal{G}_\tau^G \)-irresolute function.
2. \( g \circ f \) is \( \mathcal{G}_\rho^G \)-irresolute function if \( f \) and \( g \) are \( \mathcal{G}_\tau^G \)-irresolute functions.

Theorem 3.21. Let \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho) \) and \( g : (Y, \rho) \rightarrow (Z, \gamma) \) be two functions. Then \( g \circ f \) is \( \mathcal{G}_\tau^G \)-closed function if \( g \) is a closed function and \( f \) is \( \mathcal{G}_\tau^G \)-closed function.

Proof. Let \( U \) be \( \mathcal{G}_\tau^G \)-closed set in \( (X, \tau, \mathcal{G}) \). Since \( f \) is \( \mathcal{G}_\tau^G \)-closed function then \( f(U) \) is a closed set in \( Y \). Since \( g \) is closed function then \( g(f(U)) = (g \circ f)(U) \) is closed set in \( (Z, \gamma) \). That is, \( g \circ f \) is \( \mathcal{G}_\gamma^H \)-closed function.

Corollary 3.22. Let \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho) \) and \( g : (Y, \rho) \rightarrow (Z, \gamma) \) be two functions. Then \( g \circ f \) is \( \mathcal{G}_\tau^G \)-open function if \( g \) is a open function and \( f \) is \( \mathcal{G}_\tau^G \)-open function.
The following Example shows that the composition of two $G^ω$-continuous functions need not be $G^ω$-continuous.

**Example 3.23.** Let $f : (\mathbb{R}, \tau, G) \rightarrow (Y, \rho)$ and $g : (Y, \rho, H) \rightarrow (Y, \gamma)$ be two functions defined by:

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - (0,1) \\ 1, & x \in (0,1) \end{cases}$$

And $g$ is identity function. Where $Y = \{0,1\}$.

$$\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0,1)\}, \ G = P(\mathbb{R}) - \{\emptyset\}, \ H = P(Y) - \{\emptyset\} \text{ and } \rho = \{\emptyset, Y, \{0\}\}.$$  

And $\gamma = \{\emptyset, \{1\}, Y\}$. Then $f$ and $g$ are $G^ω$-continuous functions but $g \circ f$ is not $G^ω$-continuous function. Since $\{0\}$ is a closed set in $(Y, \gamma)$ but $f^{-1}(\{0\}) = \mathbb{R} - (0,1)$ is not $G^ω$-closed set in $(\mathbb{R}, \tau, G)$.

The following Example shows that the composition of two $f$ is $G^ω$-continuous function and $g$ is $g\omega$-continuous function need not be $G^ω$-continuous.

**Example 3.24.** In Example 3.23, we note that $f$ is $G^ω$-continuous function and $g$ is $g\omega$-continuous function, but $g \circ f$ is not $G^ω$-continuous function. Since $\{0\}$ is a closed set in $(Y, \gamma)$ but $f^{-1}(\{0\}) = \mathbb{R} - (0,1)$ is not $G^ω$-closed set in $(\mathbb{R}, \tau, G)$.

**Theorem 3.25.** Let $f : (X, \tau, G) \rightarrow (Y, \rho)$ be a $G^ω$-closed function and $A$ is $g$-closed subset in $(X, \tau, G)$. Then $f(A)$ is $r\text{rg}\omega$-closed subset in $(Y, \rho)$.

**Proof.** Let $f$ be $G^ω$-closed function and $A \subseteq X$ such that $A$ is $g$-closed set. Since $f$ is $G^ω$-closed function and $A$ is $g$-closed set. Since it is clear that every $g$-closed set is $G^ω$-closed set then $f(A)$ is a closed set in $(Y, \rho)$. Since any closed set is $r\text{rg}\omega$-closed set. Therefore $f(A)$ is $r\text{rg}\omega$-closed set in $(Y, \rho)$. □

**Theorem 3.26.** Let $f : (X, \tau, G) \rightarrow (Y, \rho)$ be a $G^ω$-continuous function and $(Y, \rho)$ is $T_{1|2}$-space and $A$ is $g$-closed subset in $(Y, \rho)$. Then $f^{-1}(A)$ is $G^ω$-closed subset in $(X, \tau, G)$.

**Proof.** Let $f$ be $G^ω$-continuous function and $(Y, \rho)$ is $T_{1|2}$-space. Let $A$ be $g$-closed subset in $(Y, \rho)$. Since $(Y, \rho)$ is $T_{1|2}$-space, then for every $g$-closed subset in $(Y, \rho)$ is $G^ω$-closed subset in $(Y, \rho)$. Since $f$ is $G^ω$-continuous function then for every $A$ closed subset in $(Y, \rho)$ implies $f^{-1}(A)$ is $G^ω$-closed subset in $(X, \tau, G)$. □

## 4  $G^ω_{rg}$-Continuous Functions

**Definition 4.1.** A function $f : (X, \tau, G) \rightarrow (Y, \rho, H)$ of a grill topological space $(X, \tau, G)$ into a grill topological space $(Y, \rho, H)$ is called $G^ω_{rg}$-continuous function if $f^{-1}(U)$ is $G^ω_{rg}$-closed set in $(X, \tau, G)$ for every $G^ω$-closed set $U$ in $(Y, \rho, H)$.

**Definition 4.2.** A function $f : (X, \tau, G) \rightarrow (Y, \rho, H)$ of a grill topological space $(X, \tau, G)$ into a grill topological space $(Y, \tau, H)$ is called $G^ω_{rg}$-irresolute if $f^{-1}(U)$ is $G^ω_{rg}$-closed set in $(X, \tau, G)$ for every $G^ω_{rg}$-closed set in $(Y, \tau, H)$.

**Remark 4.1.** Let $(X, \tau, G)$ be any grill topological space with $\tau = \{\emptyset, X, A\}$, $G = P(X) - \{\emptyset\}$, where $A$ is an uncountable set. Then any subset of $(X, \tau, G)$ is a both $G^ω_{rg}$-open set and $G^ω_{rg}$-closed set.
Example 4.1. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \tau_D, \mathcal{H})$ be a any function with 
\[
\tau = \{\emptyset, \mathbb{R}, (0, 1)\}, \quad \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.
\]
Since every $f^{-1}(U)$ is $\mathcal{G}_\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$ for every $\mathcal{G}_\omega$-closed set $U$ in $(\mathbb{R}, \tau_D, \mathcal{H})$. Then $f$ is $\mathcal{G}_\omega$-continuous function.

Remark 4.2. Let $(X, \tau, \mathcal{G})$ be any grill topological space with 
\[
\tau = \{\emptyset, X, A \cup \{A \cup B : A \subseteq B, \text{for every } i \in \mathbb{N}\}\} \quad \text{and} \quad \mathcal{G} = P(X) - \{\emptyset\}.
\]
Where $A$ be any subset of $X$. Then any subset of $(X, \tau, \mathcal{G})$ is a both $\mathcal{G}_\tau$-open set and $\mathcal{G}_\tau$-closed set.

Example 4.2. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H})$ be a any function where $(Y, \rho, \mathcal{H})$ is any grill topological space and 
\[
\tau = \{\emptyset, \mathbb{R}, (0, 1], (0, 2], (0, 7)\}, \quad \mathcal{H} = P(Y) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.
\]
Then $f$ is $\mathcal{G}_\tau$-continuous function by Remark (4.2).

It is clear that every $rg\omega$-continuous function is $\mathcal{G}_\tau$-continuous function but the converse of this fact need not be true.

Example 4.3. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$ be the identity function with 
\[
\tau = \{\emptyset, \mathbb{R}, (0, 1], \mathbb{R} - (1, 3), \mathbb{R} - (0, 3)\}, \quad \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \rho = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, \frac{1}{2})]\}
\]
We note that $f$ is $\mathcal{G}_\omega$-continuous function. Since $(0, \frac{1}{2}]$ and $\mathbb{R}$ are closed sets in $(\mathbb{R}, \rho, \mathcal{H})$. Then $f^{-1}((0, \frac{1}{2}]) = (0, \frac{1}{2}]$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are $\mathcal{G}_\omega$-closed sets but it is not $rg\omega$-continuous function. Since $(0, \frac{1}{2}]$ is closed set in $(\mathbb{R}, \rho, \mathcal{H})$ but $f^{-1}((0, \frac{1}{2}]) = (0, \frac{1}{2}]$ is not $rg\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}_\omega$-continuous function is $\mathcal{G}_\tau$-continuous function but the converse of this fact need not be true.

Example 4.4. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho, \mathcal{H})$ be the identity function with 
\[
\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0, 1)\}, \quad \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \rho = \{\emptyset, \mathbb{R}, (0, 1)\}.
\]
The function $f$ is $\mathcal{G}_\omega$-continuous, since $f^{-1}(\{\mathbb{R} - (0, 1)\}) = \mathbb{R} - (0, 1)$ and $f^{-1}(\mathbb{R}) = \mathbb{R}$ are $\mathcal{G}_\omega$-closed sets in $(\mathbb{R}, \tau, \mathcal{G})$. The function $f$ is not $\mathcal{G}_\omega$-continuous function, since $f^{-1}(\{\mathbb{R} - (0, 1)\}) = \mathbb{R} - (0, 1)$ is not $\mathcal{G}_\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}_\omega$-irresolute function is $\mathcal{G}_\tau$-continuous function but the converse of this fact need not be true.

Example 4.5. Let $f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (\mathbb{R}, \rho_I, \mathcal{H})$ be the identity function with 
\[
\tau = \{\emptyset, \mathbb{R}, (0, 1], (0, 1), \mathbb{R} - (0, 3), \mathbb{R} - (1, 3), \mathbb{R} - [1, 3)\}, \quad \mathcal{H} = P(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}.
\]
We note that $f$ is $\mathcal{G}_\omega$-continuous function but it is not $\mathcal{G}_\omega$-irresolute function. Since $(0, 1)$ is $\mathcal{G}_\omega$-closed set in $(\mathbb{R}, \rho_I, \mathcal{H})$ but $f^{-1}((0, 1)) = (0, 1)$ is not $\mathcal{G}_\omega$-closed set in $(\mathbb{R}, \tau, \mathcal{G})$.

It is clear that every $\mathcal{G}_\omega$-irresolute function is $\mathcal{G}_\tau$-irresolute function but the converse of this fact need not be true.
Example 4.6. Let \( f : (\mathbb{R}, \tau, \mathcal{G}) \to (\mathbb{R}, \rho_1, \mathcal{H}) \) be the identity function with
\[
\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (1,3)\}, \quad \mathcal{H} = \mathcal{P}(\mathbb{R}) - \{\emptyset\} \quad \text{and} \quad \mathcal{G} = \mathcal{P}(\mathbb{R}) - \{\emptyset\}.
\]
We note that \( f \) is \( \mathcal{G}^c_{\omega} \)-irresolute function but it is not \( \mathcal{G}^c_{\tau} \)-irresolute function. Since the set \( \mathbb{R} - (1,3) \) is \( \mathcal{G}^c_{\tau} \)-closed set in \( (\mathbb{R}, \rho_1, \mathcal{H}) \) but \( f^{-1}(\mathbb{R} - (1,3)) = \mathbb{R} - (1,3) \) is not \( \mathcal{G}^c_{\omega} \)-closed set in \( (\mathbb{R}, \tau, \mathcal{G}) \).

The following figure is an enlargement of fig. 2. and an theorems above we have the following figure:

**Definition 4.3.** A function \( f : (X, \tau, \mathcal{G}) \to (Y, \rho, \mathcal{H}) \) of a grill topological space \( (X, \tau, \mathcal{G}) \) into a grill topological space \( (Y, \rho, \mathcal{H}) \) is called:

1. \( \mathcal{G}^c_{\tau} \)-closed function if \( f(G) \) is a closed set in \( (Y, \rho) \) for every \( \mathcal{G}^c_{\tau} \)-closed set \( G \) in \( (X, \tau, \mathcal{G}) \).
2. \( \mathcal{G}^c_{\tau} \)-open function if \( f(G) \) is an open set in \( (Y, \rho) \) for every \( \mathcal{G}^c_{\tau} \)-open set \( G \) in \( (X, \tau, \mathcal{G}) \).
3. \( \text{pre-} \mathcal{G}^c \)-closed function if \( f(V) \) is \( \mathcal{G}^c \)-closed in \( (Y, \rho, \mathcal{H}) \) for every \( \mathcal{G}^c \)-closed subset \( V \) of \( (X, \tau, \mathcal{G}) \).

Example 4.7. In Example (4.1), we note that \( f \) is a both \( \mathcal{G}^c_{\tau} \)-closed function and \( \mathcal{G}^c_{\tau} \)-open function.

**Definition 4.4.** A grill topological space \( (X, \tau, \mathcal{G}) \) is a regular generalized \( \mathcal{G}^c \)-\( T_{1\frac{1}{2}} \)-space (simply, \( \mathcal{G}^c_{\tau} \)-\( T_{1\frac{1}{2}} \)-space) if every \( \mathcal{G}^c_{\tau} \)-closed set in \( (X, \tau, \mathcal{G}) \) is \( \mathcal{G}^c \)-closed set.

**Theorem 4.8.** Let \( f : (X, \tau, \mathcal{G}) \to (Y, \rho, \mathcal{H}) \) be a function of a grill topological space \( (X, \tau, \mathcal{G}) \) into a grill topological pace \( (Y, \rho, \mathcal{H}) \). Then \( f \) is \( \mathcal{G}^c_{\tau} \)-continuous function if \( f^{-1}(U) \) is \( \mathcal{G}^c_{\tau} \)-open set in \( (X, \tau, \mathcal{G}) \) for every \( \mathcal{G}^c \)-open set in \( (Y, \rho, \mathcal{H}) \).

**Proof.** Let \( f \) be \( \mathcal{G}^c_{\tau} \)-continuous function. Let \( U \) be \( \mathcal{G}^c \)-open set in \( (Y, \rho, \mathcal{H}) \) then \( Y - F \) is \( \mathcal{G}^c \)-closed set in \( (Y, \rho, \mathcal{H}) \). Since \( f \) is \( \mathcal{G}^c_{\tau} \)-continuous function. Then \( f^{-1}(Y - F) = X - f^{-1}(U) \) is \( \mathcal{G}^c_{\tau} \)-closed set in a grill topological space \( (X, \tau, \mathcal{G}) \). Therefore \( f^{-1}(U) \) is \( \mathcal{G}^c_{\tau} \)-open set in \( (X, \tau, \mathcal{G}) \). Hence \( f \) is \( \mathcal{G}^c_{\tau} \)-continuous function.

**Theorem 4.9.** Let \( f : (X, \tau, \mathcal{G}) \to (Y, \rho, \mathcal{I}) \) and \( g : (Y, \rho, \mathcal{H}) \to (Z, \gamma, \mathcal{K}) \) two functions. Then the following hold:
1. \( g \circ f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function if \( g \) is \( \mathcal{G}^\omega \)-irresolute function and \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

2. \( g \circ f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function if \( g \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function and \( f \) is \( \mathcal{G}^\omega \)-irresolute function.

3. \( g \circ f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-irresolute function if \( f \) and \( g \) are \( \mathcal{G}^\omega_{\tau_Y} \)-irresolute functions.

4. Let \( (Y, \rho) \) be \( \mathcal{G}^\omega_{\tau_Y} \)-\( T_{1/2} \)-space. Then, \( g \circ f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function if \( g \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function and \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

The following Example shows that the composition of two \( \mathcal{G}^\omega_{\tau_Y} \)-continuous functions need not be \( \mathcal{G}^\omega_{\tau_Y} \)-continuous.

**Example 4.10.** Let \( f : (\mathbb{R}, \tau, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H}) \) and \( g : (Y, \rho, \mathcal{H}) \rightarrow (Y, \gamma, \mathcal{K}) \) be two functions defined by:

\[
  f(x) = \begin{cases} 
    0, & x \in (0,1) \\
    1, & x \in \mathbb{R} - (0,1)
  \end{cases}
\]

And \( g \) is identity function. Where \( Y = \{0,1\}, \)

\[
  \tau = \{\emptyset, (0,1), (0,1), \mathbb{R} - (0,3), \mathbb{R} - (1,3), \mathbb{R} - [1,3]\}, \ G = P(\mathbb{R}) - \{\emptyset\}, \ K = P(Y) - \{\emptyset\}, \ H = P(Y) - \{\emptyset\},
\]

\[
  \rho = \{\emptyset, Y, \{0\}\} \text{ and } \gamma = \{\emptyset, \{1\}, Y\}.
\]

Then \( f \) and \( g \) are \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function but \( g \circ f \) is not \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function. Since \( \{0\} \) is \( \mathcal{G}^\omega \)-closed set in \((Y, \gamma, \mathcal{K})\) but \( f^{-1}(\{0\}) = (0,1) \) is not \( \mathcal{G}^\omega_{\tau_Y} \)-closed set in \((\mathbb{R}, \tau, \mathcal{G})\).

**Theorem 4.11.** Let \((X, \tau)\) be any door topological space and \( f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho_D, \mathcal{H}) \) be any function. Then \( f \) is both \( \mathcal{G}^\omega_{\tau_Y} \)-open function and \( \mathcal{G}^\omega_{\tau_Y} \)-closed function and it is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

**Proof.** Let \((X, \tau)\) be any door topological space. Let \( A \subseteq X \), since \( X \) is a door topological space, then \( A \) is an either open set or a closed set. If \( A \) is an open set of \( X \), it is clear that every open set is \( \mathcal{G}^\omega_{\tau_Y} \)-open set implies \( A \) is \( \mathcal{G}^\omega_{\tau_Y} \)-open set. Then \( f(A) \) is an open set in \( Y \). Therefore \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-open function. If \( A \) is a closed set of \( X \), it clear that every closed set is \( \mathcal{G}^\omega_{\tau_Y} \)-closed set implies \( A \) is \( \mathcal{G}^\omega_{\tau_Y} \)-closed set. Then \( f(A) \) is a closed set in \( Y \). Therefore \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-closed function.

Let \( A \subseteq Y \), since \( X \) is a door topological space then \( f^{-1}(A) \) is an either open set or a closed set in \( X \). If \( f^{-1}(A) \) is \( \mathcal{G}^\omega_{\tau_Y} \)-open set, since \( Y \) is \((Y, \rho_D)\) then \( A \) is an open set. If \( f^{-1}(A) \) is \( \mathcal{G}^\omega_{\tau_Y} \)-closed set since \( Y \) is \((Y, \rho_D)\) then \( A \) is a closed set. Therefore \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

In Theorem (4.11), if \( X \) is a countable set and \((Y, \rho)\) be any topological space then \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

**Theorem 4.12.** Let \( f : (X, \tau_Y, \mathcal{G}) \rightarrow (Y, \rho_D, \mathcal{H}) \) be any function. Then \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

**Proof.** Let \( A \subseteq Y \) and since \( Y \) is \((Y, \rho_D, \mathcal{H})\), then any subset of \( Y \) is a both open set and closed set. Now \( f^{-1}(A) \) is a both \( \mathcal{G}^\omega_{\tau_Y} \)-open set and \( \mathcal{G}^\omega_{\tau_Y} \)-closed set. Therefore \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function by Remark(3.1).

**Theorem 4.13.** Let \( f : (X, \tau_Y, \mathcal{G}) \rightarrow (Y, \rho, \mathcal{H}) \) be any function. Then \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.

**Proof.** Let \( A \) be an open subset of \( Y \). Then \( f^{-1}(A) \) is \( \mathcal{G}^\omega_{\tau_Y} \)-open set by Remark(3.1). Let \( A \) be a closed subset of \( Y \). Then \( f^{-1}(A) \) is \( \mathcal{G}^\omega_{\tau_Y} \)-closed set by Remark(3.1). Therefore \( f \) is \( \mathcal{G}^\omega_{\tau_Y} \)-continuous function.
Theorem 4.14. Every $G_{rg}^\omega$-closed function is $G^\omega_g$-closed function but the converse of this fact need not be true.

Theorem 4.15. A grill topological space $(X, \tau, G)$ is called a regular generalized $G^\omega - T_{1\frac{1}{2}}$-space (simply $G^\omega_{rg} - T_{1\frac{1}{2}}$-space) if and only if every $G_{rg}^\omega$-open set in $(X, \tau, G)$ is $G^\omega$-open set.

Theorem 4.16. A grill topological space $(X, \tau, G)$ is called a regular generalized $G^\omega - T_{1\frac{1}{2}}$-space (simply $G_{rg}^\omega - T_{1\frac{1}{2}}$-space) if and only if every singleton set in $(X, \tau, G)$ is either regular-closed set or $G^\omega$-open set.

Proof. Suppose that $(X, \tau, G)$ is $G^\omega_{rg} - T_{1\frac{1}{2}}$-space and $\{x\}$ is not regular-closed subset of $X$ for some $x \in X$. Then $X - \{x\}$ is not regular-open set in $X$. Hence $X$ is the only regular-open set containing $X - \{x\}$. That is, $X - \{x\}$ is $G^\omega_{rg}$-closed set in $X$. Since $(X, \tau, G)$ is $G^\omega_{rg} - T_{1\frac{1}{2}}$-space then $X - \{x\}$ is $G^\omega$-closed set in $X$. That is, $\{x\}$ is a $G^\omega$-open set in $X$.

Conversely, suppose that every singleton set in $X$ is either regular-closed or $G^\omega$-open Let $A$ be any $G^\omega_{rg}$-closed set in $X$ and $x \in \varnothing - Cl(A)$. We show that $x \in A$. By the hypothesis $\{x\}$ is either regular-closed set or $G^\omega$-open set in $X$. The set $\{x\}$ is regular-closed set and $x \notin A$ then

$$x \in \varnothing - Cl(A) - A \subseteq X - A.$$  

Then $\{x\} \subseteq X - A$ and so $A \subseteq X - \{x\}$. Since $A$ is $G^\omega_{rg}$-closed set in $X$ contained in regular-open set $X - \{x\}$ then $\varnothing - Cl(A) \subseteq X - \{x\}$ and so $\{x\} \subseteq X - \varnothing - Cl(A)$. Therefore

$$\{x\} \subseteq \varnothing - Cl(A) \cap [X - \varnothing - Cl(A)] = 0,$$

and this is a contradiction. Hence $\{x\} \in A$, that is, $\varnothing - Cl(A) = A$ and so $A$ is $G^\omega$-closed set. If $\{x\}$ is $G^\omega$-open set and $\{x\} \in \varnothing - Cl(A)$ then we have $\{x\} \cap A \neq \emptyset$. Hence $\{x\} \in A$, that is, $\varnothing - Cl(A) = A$ and so $A$ is $G^\omega$-closed set.

Theorem 4.17. If $f : (X, \tau, G) \rightarrow (Y, \rho, H)$ is $G^\omega_{rg}$-continuous function and $X$ is $G^\omega_{rg} - T_{1\frac{1}{2}}$-space, then $f$ is $G^\omega$-continuous function.

Proof. Let $f$ be any $G^\omega_{rg}$-continuous function and $X$ is $G^\omega_{rg} - T_{1\frac{1}{2}}$-space, then every $G^\omega_{rg}$-closed subset of $X$ is $G^\omega$-closed set. Let $A$ be any closed set in $Y$. Since any closed set is $G^\omega$-closed set, $f$ is $G^\omega_{rg}$-continuous function and $G^\omega_{rg} - T_{1\frac{1}{2}}$-space, then $f^{-1}(A)$ is $G^\omega$-closed set. Therefore $f$ is $G^\omega$-continuous function.

Theorem 4.18. Let $(X, \tau, G)$ be any grill topological space with a countable set $X$. Then every $G^\omega_{rg}$-closed subset of $X$ is $G^\omega_g$-closed set.

Theorem 4.19. Let $f : (X, \tau, G) \rightarrow (Y, \rho, H)$ be any function and $X$ is countable set. Then $f$ is $G^\omega_{rg}$-continuous function and it is $G^\omega_g$-continuous function.

Theorem 4.20. If $f : (X, \tau, G) \rightarrow (Y, \rho, H)$ is $G^\omega_{rg}$-continuous, then for each $x \in X$ and each $G^\omega$-pen set $V$ in $(Y, \rho, H)$ with $f(x) \in V$, there exists $G^\omega_{rg}$-open set $U$ in $(X, \tau, G)$ such that $x \in U$ and $f(U) \subseteq V$.

Proof. Let $f$ is $G^\omega_{rg}$-continuous function. Let $x \in X$ and $V$ be any $G^\omega$-open set in $(Y, \rho, H)$ containing $f(x)$. Take $U = f^{-1}(V)$. Since $f$ is $G^\omega_{rg}$-continuous then $U$ is $G^\omega_{rg}$-open set in $(X, \tau, G)$ such that $x \in U$ and $f(U) \subseteq V$. The converse of the above Theorem is not true in general as the following Example shows.
Example 4.21. In Example (4.10), we note that \( f \) is satisfies the property stated in Theorem (4.20). The function \( f \) is not \( \mathcal{G}^\omega_{rg} \)-continuous function. Since \( \{0\} \) is \( \mathcal{G}^\omega \)-closed set in \( Y \) but \( f^{-1}(\{0\}) = (0,1) \) is not \( \mathcal{G}^\omega_{rg} \)-closed set in \( (\mathbb{R}, \tau, \mathcal{G}) \).

Theorem 4.22. Let \( f : (X, \tau, \mathcal{G}) \to (Y, \rho, \mathcal{H}) \) be a surjective, \( \mathcal{G}^\omega_{rg} \)-irresolute, and pre-\( \mathcal{G}^\omega \)-closed function. If \( X \) is \( \mathcal{G}^\omega_{rg} \)-T\(_{1/2}\)-space, then \( Y \) is also an \( \mathcal{G}^\omega_{rg} \)-T\(_{1/2}\)-space.

Proof. Let \( A \) be \( \mathcal{G}^\omega_{rg} \)-closed subset of \( Y \). Since \( f \) is an \( \mathcal{G}^\omega_{rg} \)-irresolute function, then \( f^{-1}(A) \) is an \( \mathcal{G}^\omega_{rg} \)-closed subset of \( X \). Since \( X \) is \( \mathcal{G}^\omega_{rg} \)-T\(_{1/2}\)-space, then \( f^{-1}(A) \) is an \( \mathcal{G}^\omega \)-closed subset of \( X \). Since \( f \) is a pre-\( \mathcal{G}^\omega \)-closed function, then \( f(f^{-1}(A)) = A \) is an \( \mathcal{G}^\omega \)-closed subset of \( Y \). Therefore \( Y \) is also \( \mathcal{G}^\omega_{rg} \)-T\(_{1/2}\)-space.

Theorem 4.23. Let \( f : (X, \tau, \mathcal{G}) \to (Y, \rho, \mathcal{H}) \) be \( \mathcal{G}^\omega \)-preserving and \( \mathcal{G}^\omega \)-irresolute function, if \( B \) is \( \mathcal{G}^\omega_{rg} \)-closed set in \( X \), then \( f^{-1}(B) \) is \( \mathcal{G}^\omega_{rg} \)-closed set in \( X \).

Proof. Let \( U \) be a regular-open subset of \( X \) such that \( f^{-1}(B) \subseteq U \). Then \( B \subseteq f(U) \) and \( f(U) \) is regular-open. Since \( B \) is \( \mathcal{G}^\omega_{rg} \)-closed set, then \( \mathcal{G}_\omega \text{-Cl}(B) \subseteq f(U) \) and \( f^{-1}(\mathcal{G}_\omega \text{-Cl}(B)) \subseteq U \). Since \( f \) is \( \mathcal{G}^\omega \)-irresolute then \( f^{-1}(\mathcal{G}_\omega \text{-Cl}(B)) \) is \( \mathcal{G}^\omega \)-closed and \( \mathcal{G}_\omega \text{-Cl}(f^{-1}(\mathcal{G}_\omega \text{-Cl}(B))) = f^{-1}(\mathcal{G}_\omega \text{-Cl}(B)) \), therefore \( \mathcal{G}_\omega \text{-Cl}(f^{-1}(B)) \subseteq \mathcal{G}_\omega \text{-Cl}(f^{-1}(\mathcal{G}_\omega \text{-Cl}(B))) \subseteq U \) thus \( f^{-1}(B) \) is \( \mathcal{G}^\omega_{rg} \)-closed set in \( X \).

Definition 4.5. A function \( f : (X, \tau, \mathcal{G}) \to (Y, \rho) \) is said to be \( \mathcal{G}^\omega \)-contra-\( R \)-function if for every regular-open subset \( V \) of \( Y \), \( f^{-1}(V) \) is \( \mathcal{G}^\omega \)-closed set.

Example 4.24. Let \( f : (\mathbb{R}, \tau, \mathcal{G}) \to (Y, \rho) \) be function defined by:

\[
f(x) = \begin{cases} 
  c, & \text{if } x \in \mathbb{R} - (0,1] \\
  a, & \text{if } x \in (0,1]
\end{cases}
\]

where \( Y = \{a,b,c\} \),

\[
\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - (0,1], \mathcal{G} = P(\mathbb{R}) - \{\emptyset\}, \text{ and } \rho = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.
\]

Then \( f \) is \( \mathcal{G}^\omega \)-contra-\( R \)-function. But the function \( f(x) \) defined by:

\[
f(x) = \begin{cases} 
  b, & \text{if } x \in \mathbb{R} - (0,1) \\
  a, & \text{if } x \in (0,1]
\end{cases}
\]

is not \( \mathcal{G}^\omega \)-contra-\( R \)-function. Since \( \{b\} \) is regular-open in \( Y \) and \( f^{-1}(\{b\}) = \mathbb{R} - (0,1] \) is not \( \mathcal{G}^\omega \)-closed.

Theorem 4.25. Let \( f : (X, \tau, \mathcal{G}) \to (Y, \rho) \) be \( \mathcal{G}^\omega_{rg} \)-closed function. Then every \( U \) is \( \mathcal{G}^\omega_{rg} \)-closed set in \( X \) exists \( V \) is closed set in \( Y \) such that \( f^{-1}(V) \in \mathcal{G}^\omega_{rg} \text{-Cl}(X, \tau, \mathcal{G}) \).

Proof. Let \( U \) be any \( \mathcal{G}^\omega_{rg} \)-closed set in \( (X, \tau, \mathcal{G}) \). Since \( f \) is \( \mathcal{G}^\omega_{rg} \)-closed function. Then every \( \mathcal{G}^\omega_{rg} \)-closed set in \( (X, \tau, \mathcal{G}) \) such that \( f(U) = V \) is a closed set in \( (Y, \rho) \). Therefore \( f^{-1}(V) = U \) is \( \mathcal{G}^\omega_{rg} \)-closed set.

5 Conclusions

From this papers, we conclude that \( \mathcal{G}^\omega \)-continuous functions in any grill topological space \( (X, \tau, \mathcal{G}) \) are weaker than \( \mathcal{G}^\omega_{rg} \)-continuous functions in a topological space \( (X, \tau) \), we conclude that \( \mathcal{G}^\omega_{rg} \)-continuous functions in any grill topological space \( (X, \tau, \mathcal{G}) \) are weaker than \( \mathcal{G}^\omega_{rg} \)-continuous functions in a topological space \( (X, \tau) \), we conclude that \( \mathcal{G}^\omega \)-irresolute functions, \( \mathcal{G}^\omega_{rg} \)-irresolute functions and \( \mathcal{G}^\omega_{rg} \)-irresolute functions in any grill topological space \( (X, \tau, \mathcal{G}) \) are stronger than \( \mathcal{G}^\omega \)-continuous
functions, $G_\omega^r$—continuous functions and $G_{\omega g}^r$—continuous functions, respectively in grill topological space $(X, \tau, G)$.

**Acknowledgement**

I would like to extend my thanks and gratitude to my principal supervisor, Prof. Dr. Amin Hamoud Ahmed Saif Sanawy, Assoc. Prof. in algebraic topology.

**Competing Interests**

Authors have declared that no competing interests exist.

**References**


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Peer-review history: The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar) https://www.sdiarticle5.com/review-history/88413